

REAL HYPERBOLIC HYPERPLANE COMPLEMENTS IN THE COMPLEX HYPERBOLIC PLANE

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ABSTRACT. This paper studies Riemannian manifolds of the form $M \setminus S$, where M^4 is a complete four dimensional Riemannian manifold with finite volume whose metric is modeled on the complex hyperbolic plane \mathbb{CH}^2 , and S is a compact totally geodesic codimension two submanifold whose induced Riemannian metric is modeled on the real hyperbolic plane \mathbb{H}^2 . In this paper we write the metric on \mathbb{CH}^2 in polar coordinates about S , compute formulas for the components of the curvature tensor in terms of arbitrary warping functions (Theorem 7.1), and prove that there exist warping functions that yield a complete finite volume Riemannian metric on $M \setminus S$ whose sectional curvature is bounded above by a negative constant (Theorem 1.1(1)). The cases of $M \setminus S$ modeled on $\mathbb{H}^n \setminus \mathbb{H}^{n-2}$ and $\mathbb{CH}^n \setminus \mathbb{CH}^{n-1}$ were studied by Belegarde in [Bel12] and [Bel11], respectively. One may consider this work as “part 3” to this sequence of papers.

1. INTRODUCTION

Let M be a complete (connected) locally symmetric Riemannian manifold with finite volume and negative sectional curvature, and let S be a (possibly disconnected) compact totally geodesic codimension two submanifold of M . It is known that the pair (M, S) is modeled on $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$, or the “exceptional case” $(\mathbb{CH}^2, \mathbb{H}^2)$. In [Bel12] and [Bel11] Belegarde provides an in depth study of $M \setminus S$, the manifold obtained from M by “drilling out” S , when the model for the pair (M, S) is one of the first two situations. Here we consider the exceptional case, when (M, S) is modeled on $(\mathbb{CH}^2, \mathbb{H}^2)$.

The main result proved in this paper is the following.

Theorem 1.1. *If M is a complete finite volume complex hyperbolic 2-manifold and S is a compact totally real totally geodesic 2-dimensional submanifold, then*

- (1) *$M \setminus S$ admits a complete finite volume metric with sectional curvature ≤ -1 .*
- (2) *$M \setminus S$ admits a complete finite volume A-regular metric with sectional curvature < 0 .*

The manifold $M \setminus S$ is diffeomorphic to a compact manifold N obtained by cutting out a tubular neighborhood of S in M and removing all cusps in the case that M is not compact. There are two possible types of boundary components of N . The first are compact infranil manifolds, which arise as cross-sections of the

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cusps of M (if any), and the second type is circle bundles over the components of S .

All three statements in the following Corollary 1.2 can be deduced from other known results, as will be discussed in Remark 1.3. But combining Theorem 1.1 with the methods deployed in [Bel12] and [Bel11] provides independent proofs of these statements.

Corollary 1.2. *Suppose that M is a complete finite volume complex hyperbolic 2-manifold and S is a compact totally real totally geodesic 2-dimensional submanifold. Then*

- (1) *the group $\pi_1(M \setminus S)$ is non-elementary (strongly) relatively hyperbolic, where the peripheral subgroups are the fundamental groups of the ends of $M \setminus S$.*
- (2) *$\pi_1(M \setminus S)$ satisfies the Farrell-Jones isomorphism conjecture.*
- (3) *$\pi_1(M \setminus S)$ satisfies the Rapid Decay Property and the Baum-Connes conjecture.*

Remark 1.3. The proof that Corollary 1.2(1) follows from Theorem 1.1(1) is identical to its analogues in [Bel11] and [Bel12]. In particular, see Section 12 of [Bel11] in conjunction with Theorem 4.2 in [Bel12]. Many other properties of $\pi_1(M \setminus S)$ are then known to follow from Corollary 1.2(1) (see most conclusions of Theorem 1.1 in [Bel12] and Theorem 1.4 in [Bel11]). But the fact that $\pi_1(M \setminus S)$ is relatively hyperbolic relative to the fundamental groups of its ends in our situation of (M, S) being modeled on $(\mathbb{CH}^2, \mathbb{H}^2)$ is now a special case of Corollary 1.2 by Belegradek and Hruska in [BH13].

Results by Roushon ([Rou08a] and [Rou08b]) together with a deep result of Bartels-Farrell-Lück ([BFL14]) imply that the fundamental groups of circle bundles over closed hyperbolic surfaces satisfy the Farrell-Jones isomorphism conjecture ([FJ93], abbreviated FJIC). A recent preprint by Bartels [Bar16] proves that, if a countable group G is relatively hyperbolic relative to a collection of subgroups that all satisfy the FJIC, then G satisfies the FJIC. So combining Corollary 1.2(1) with Bartels' preprint proves Corollary 1.2(2).

We can deduce the Rapid Decay Property (RDP) as follows. Drutu and Sapir [DS05] proved that if a finitely generated group is relatively hyperbolic relative to subgroups P_1, \dots, P_n , then G satisfies the RDP if and only if all of the subgroups P_1, \dots, P_n do. So it remains to check the RDP for the fundamental groups of the ends of $M \setminus S$. But the ends of $M \setminus S$ are either infranil manifolds or circle bundles over components of S . In the first case, the RDP was established by Jolissaint in [Jol90], and in the second case it is established by Garncarek in [Gar15] (also, one could combine [Jol90] with [Nos92]). Finally, Lafforgue [Laf02] proved that the fundamental group of any complete Riemannian manifold equipped with a non-positively curved A -regular metric which satisfies the RDP must also satisfy the Baum-Connes conjecture.

Theorem 1.1(1) is proved in Sections 2 through 11, an outline of which is as follows. Let \mathbb{H}^2 denote a totally real totally geodesic 2-plane in the complex hyperbolic plane \mathbb{CH}^2 . We first express the metric in \mathbb{CH}^2 in polar coordinates about \mathbb{H}^2 (Section 2, or see below for the formula). We then, allowing for different coefficient functions in this metric, derive formulas for the components of the curvature tensor

(Sections 3 to 7, Theorem 7.1). Our approach for Theorem 7.1 is direct. We first fix a “nice” non-holonomic frame of $\mathbb{CH}^2 \setminus \mathbb{H}^2$, compute the Lie brackets of these vector fields, compute the Levi-Civita connection with respect to this frame, and finally we compute the components of the curvature tensor. By far, the most difficult part of this is computing the Lie brackets (Theorem 5.2). In Sections 8 and 9 we prove a few general Lemma’s about the sectional curvature functional. Finally, we construct specific “warping” functions and prove that the sectional curvature of this warped metric is bounded above by a negative constant (Sections 10 and 11, which occupy the majority of this manuscript).

Various types of warped metric computations have been used to prove some very important results in Riemannian geometry. Recently, they have been used by Ontaneda to prove the existence of “smooth Riemannian hyperbolization” ([Ont14]). A simple warped product was used by Gromov and Thurston in [GT87] in a key way, while more difficult warped product computations have been utilized by Farrell and Jones, or Farrell and Ontaneda, in a variety of papers (see [Ont15] and the references therein). Therefore, the formulas in Theorem 7.1 for the components of the curvature tensor of our warped metric may be the most useful portion of this paper. So let us describe them.

Let r denote the distance from a point to the totally real totally geodesic 2-plane \mathbb{H}^2 , and let $\frac{\partial}{\partial r}$ be the unit length vector field on $\mathbb{CH}^2 \setminus \mathbb{H}^2$ pointing radially from \mathbb{H}^2 . Then there exists an orthogonal, non-holonomic set of vector fields (S, T) on $\mathbb{CH}^2 \setminus \mathbb{H}^2$ such that the complex hyperbolic metric g is

$$g = \cosh^2\left(\frac{r}{2}\right) dS^2 + \cosh^2(r) dT^2 + \sinh^2\left(\frac{r}{2}\right) d\theta^2 + dr^2.$$

In the above formula, dS and dT denote the covector fields dual to S and T , respectively, and $d\theta$ denotes the standard metric on the unit circle. See Sections 2 through 7 for more details, especially about S and T . Let

$$\lambda = h_\theta^2(r) dS^2 + h_r^2(r) dT^2 + v^2(r) d\theta^2 + dr^2$$

where h_θ , h_r , and v are positive functions of r . Let $Y_1 = \frac{1}{v} \frac{\partial}{\partial \theta}$, $Y_2 = \frac{1}{h_\theta} S$, $Y_3 = \frac{1}{h_r} T$, and $Y_4 = \frac{\partial}{\partial r}$ be a λ -orthonormal frame of $\mathbb{CH}^2 \setminus \mathbb{H}^2$. Then the formulas for the components of the curvature tensor of λ with respect to h_θ , h_r , and v are

$$\begin{aligned} \langle R_\lambda(Y_1, Y_2)Y_1, Y_2 \rangle_\lambda &= -\frac{v'h'_\theta}{vh_\theta} - \frac{1}{4} \left(\frac{-v^2}{4h_\theta^2 h_r^2} - \frac{h_\theta^2}{4v^2 h_r^2} + \frac{3h_r^2}{4v^2 h_\theta^2} - \frac{1}{2v^2} + \frac{1}{2h_\theta^2} - \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_1, Y_3)Y_1, Y_3 \rangle_\lambda &= -\frac{v'h'_r}{vh_r} - \frac{1}{4} \left(\frac{-v^2}{4h_\theta^2 h_r^2} + \frac{3h_\theta^2}{4v^2 h_r^2} - \frac{h_r^2}{4v^2 h_\theta^2} - \frac{1}{2v^2} - \frac{1}{2h_\theta^2} + \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_2, Y_3)Y_2, Y_3 \rangle_\lambda &= -\frac{h'_\theta h'_r}{h_\theta h_r} - \frac{1}{4} \left(\frac{3v^2}{4h_\theta^2 h_r^2} - \frac{h_\theta^2}{4v^2 h_r^2} - \frac{h_r^2}{4v^2 h_\theta^2} + \frac{1}{2v^2} + \frac{1}{2h_\theta^2} + \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_1, Y_2)Y_3, Y_4 \rangle_\lambda &= \frac{-1}{4h_r} \left[\left(\frac{h_\theta}{v} \right)' - \left(\frac{v}{h_\theta} \right)' - \left(\frac{h_r^2}{vh_\theta} \right)' \right] \\ \langle R_\lambda(Y_1, Y_3)Y_2, Y_4 \rangle_\lambda &= \frac{-1}{4h_\theta} \left[-\left(\frac{h_r}{v} \right)' + \left(\frac{v}{h_r} \right)' + \left(\frac{h_\theta^2}{vh_r} \right)' \right] \end{aligned}$$

$$\begin{aligned}
\langle R_\lambda(Y_1, Y_4)Y_2, Y_3 \rangle_\lambda &= \frac{-1}{4v} \left[\left(\frac{h_\theta}{h_r} \right)' + \left(\frac{h_r}{h_\theta} \right)' + \left(\frac{v^2}{h_\theta h_r} \right)' \right] \\
\langle R_\lambda(Y_1, Y_4)Y_1, Y_4 \rangle_\lambda &= -\frac{v''}{v} & \langle R_\lambda(Y_2, Y_4)Y_2, Y_4 \rangle_\lambda &= -\frac{h_\theta''}{h_\theta} \\
\langle R_\lambda(Y_3, Y_4)Y_3, Y_4 \rangle_\lambda &= -\frac{h_r''}{h_r}
\end{aligned}$$

and where all other components of the curvature tensor are identically zero. Of course, the formulas above intimately depend on the vector fields S and T (which are parallel to $J \frac{\partial}{\partial \theta}$ and $J \frac{\partial}{\partial r}$, respectively). These vector fields are constructed in Section 2.

We end this paper by proving Theorem 1.1(2) in Section 12. This construction and proof are nearly identical to Section 11 of [Bel11], and we only include some details here due to the slight differences in both our metric and curvature formulas (see Section 8). In fact, there are several places in this paper where we refer to results in [Bel11], and others where we follow [Bel11] very closely. So the reader interested in understanding every detail of these results should also have a copy of [Bel11] at hand. In fact, one may consider this paper as “part 3” in the sequence of papers [Bel12] and [Bel11], or maybe more accurately just the sequel to [Bel11]. Many of the situations that arise in this work are considerably different than those in [Bel11], but whenever possible we have tried to use results from Belegradek’s paper in order to simplify calculations here. Lastly, we have in most cases tried to use the same notation as [Bel11] in order to make it easier to simultaneously read the two papers.

Remark 1.4. One major notational difference between this paper and the papers [Bel11] and [Bel12] is the following. Let g be a Riemannian metric with Levi-Civita connection ∇ , and let W, X, Y , and Z be vector fields. In this paper we use the notation

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$$

for the curvature tensor R of g . The negative of this formula is used in [Bel11] and [Bel12]. So, in particular, the $(4, 0)$ -curvature tensor $\langle R(X, Y)Z, W \rangle_g$ in this paper is equivalent to $\langle R(X, Y)W, Z \rangle_g$ in [Bel11] and [Bel12].

2. THE METRIC IN COMPLEX HYPERBOLIC 2-SPACE IN CYLINDRICAL COORDINATES ABOUT A TOTALLY REAL TOTALLY GEODESIC 2-PLANE

The purpose of this section is to describe the metric for \mathbb{CH}^2 in cylindrical coordinates about a totally real totally geodesic 2-plane denoted by \mathbb{H}^2 . The case of \mathbb{CH}^n about \mathbb{CH}^{n-1} was worked out in [Bel11], and the metric on distance spheres in \mathbb{CH}^n is discussed by Farrell and Jones in [FJ94]. The current case is worked out by Phillips in an undergraduate REU [Phi08], but in the following sections we will rely on the vector fields S and T defined below in a very crucial way. So what follows is a detailed explanation of the construction in [Phi08]. The terminology and notation in this section will closely follow [Bel11] and [Bel12].

Let g denote the complex hyperbolic metric on \mathbb{CH}^2 normalized to have constant holomorphic sectional curvature -1 . Since \mathbb{H}^2 is a complete totally geodesic

submanifold of the negatively curved manifold \mathbb{CH}^2 , there exists an orthogonal projection map $\pi : \mathbb{CH}^2 \rightarrow \mathbb{H}^2$. This map π is a fiber bundle whose fibers are totally real totally geodesic 2-planes, and therefore have constant sectional curvature $-\frac{1}{4}$.

For $r > 0$ let $F(r)$ denote the r -neighborhood of \mathbb{H}^2 . Then $F(r)$ is a real hypersurface in \mathbb{CH}^2 , and consequently we can decompose g as

$$g = dr^2 + g_r$$

where g_r is the induced Riemannian metric on $F(r)$. Let $\pi_r : F(r) \rightarrow \mathbb{H}^2$ denote the restriction of π to $F(r)$. Note that π_r is a circle bundle whose fiber over any point $q \in \mathbb{CH}^2$ is the circle of radius r in the totally real totally geodesic 2-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the circle $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

For $r, v > 0$ there exists a diffeomorphism $\phi_{vr} : F(v) \rightarrow F(r)$ induced by the geodesic flow along the totally real totally geodesic 2-planes orthogonal to \mathbb{H}^2 . Fix $p \in F(r)$ arbitrary, let $q = \pi(p) \in \mathbb{H}^2$, and let γ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_p F(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $F(r)$ and the totally real totally geodesic 2-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{vr}$ takes $\mathcal{V}(v)$ to $\mathcal{V}(r)$. Since $\exp_p^{-1}(\pi^{-1}(q))$ is a real 2-plane with curvature $-\frac{1}{4}$, the metric g restricted to $\pi^{-1}(q)$ can be written as $dr^2 + \sinh^2\left(\frac{r}{2}\right) d\theta^2$ where $d\theta^2$ is the standard metric on the unit circle. Note that the vector field $\frac{\partial}{\partial \theta}$ is invariant under $d\phi_{vr}$.

Let J denote the complex structure on \mathbb{CH}^2 . It is well known that J_p preserves complex lines in $T_p \mathbb{CH}^2$ and maps real 2-planes into (and, via dimension reasons in our setting, onto) their orthogonal complement. Since $(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial r})$ spans a real 2-plane in $T_p \mathbb{CH}^2$, their orthogonal complement $\mathcal{H}_p(r)$ is spanned by $(J\frac{\partial}{\partial \theta}, J\frac{\partial}{\partial r})$. In the following two Subsections we define vector fields S and T which are just scaled copies of $J\frac{\partial}{\partial \theta}$ and $J\frac{\partial}{\partial r}$, respectively.

2.1. Definition of the vector field S . First note that $(\frac{\partial}{\partial r}, J\frac{\partial}{\partial \theta})$ spans a real 2-plane in $T_p \mathbb{CH}^2$ (since its J -image is its orthogonal complement). So $P = \exp(\text{span}(\frac{\partial}{\partial r}, J\frac{\partial}{\partial \theta}))$ is a totally real totally geodesic 2-plane in \mathbb{CH}^2 which intersects \mathbb{H}^2 orthogonally. Since this intersection is orthogonal, P is preserved by the geodesic flow ϕ . Therefore, $\text{span}(J\frac{\partial}{\partial \theta})$ is preserved by $d\phi$.

The set $P \cap \mathbb{H}^2$ is a (real) geodesic. Let $\alpha(s)$ denote this geodesic parameterized with respect to arc length so that $\alpha(0) = q$. Then define $S_p = (d\pi)_p^{-1} \alpha'(0)$. There exists a positive real-valued function $a(r, s)$ so that the metric g restricted to P is of the form $dr^2 + a^2(r, s) ds^2$. But note that $a(r, s)$ is independent of s because of the isometric \mathbb{R} -action on P by translations along α . Then since the curvature of a real 2-plane is $-\frac{1}{4}$, we have that $a(r) = \cosh\left(\frac{r}{2}\right)$.

2.2. Definition of the vector field T . This is analogous to the definition of S . Note that $(\frac{\partial}{\partial r}, J\frac{\partial}{\partial r})$ spans a complex line in $T_p \mathbb{CH}^2$ (since it is preserved by its J -image). So $Q = \exp(\text{span}(\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}))$ is a complex geodesic in \mathbb{CH}^2 which

intersects \mathbb{H}^2 orthogonally. Since this intersection is orthogonal, Q is preserved by the geodesic flow ϕ . Therefore, $\text{span}\left(J\frac{\partial}{\partial r}\right)$ is preserved by $d\phi$.

The set $Q \cap \mathbb{H}^2$ is a (real) geodesic. Let $\beta(t)$ denote this geodesic parameterized with respect to arc length so that $\beta(0) = q$. Then define $T_p = (d\pi)^{-1}\beta'(0)$. There exists a positive real-valued function $b(r, t)$ so that the metric g restricted to Q is of the form $dr^2 + b^2(r, t)dt^2$. But note that $b(r, t)$ is independent of t because of the isometric \mathbb{R} -action on Q by translations along β . Then since the curvature of a complex geodesic is -1 , we have that $b(r) = \cosh r$.

2.3. Conclusion.

Theorem 2.1. *The complex hyperbolic manifold $\mathbb{CH}^2 \setminus \mathbb{H}^2$ can be written as $(0, \infty) \times F$ where $F \cong \mathbb{R}^2 \times \mathbb{S}^1$ equipped with the metric*

$$(2.1) \quad g = \sinh^2\left(\frac{r}{2}\right) d\theta^2 + \cosh^2\left(\frac{r}{2}\right) dS^2 + \cosh^2(r) dT^2 + dr^2.$$

In equation (2.1), dS and dT denote the covector fields dual to the vector fields S and T , respectively. Lastly, notice that $dS^2 + dT^2$ is the hyperbolic metric with constant sectional curvature $-\frac{1}{4}$.

3. THE METRIC λ AND LIE BRACKETS (PART I)

Fix an open interval I and let v , h_θ , and h_r be smooth positive functions on I . Let λ denote the Riemannian metric

$$\lambda = v^2 d\theta^2 + h_\theta^2 dS^2 + h_r^2 dT^2 + dr^2$$

on $I \times F$ (and where $F = F(r)$ for some generic $r > 0$). Of course, we recover the metric on $\mathbb{CH}^2 \setminus \mathbb{H}^2$ when $v = \sinh\left(\frac{r}{2}\right)$, $h_\theta = \cosh\left(\frac{r}{2}\right)$, and $h_r = \cosh r$. The purpose of Sections 3 through 7 is to compute the components of the curvature tensor R_λ in terms of v , h_λ , and h_r .

For these curvature computations we will use the non-holonomic basis

$$(3.1) \quad X_1 = \frac{\partial}{\partial \theta}, \quad X_2 = S, \quad X_3 = T, \quad X_4 = \frac{\partial}{\partial r}.$$

Let us note the following observations about this basis:

- (1) $\langle X_1, X_1 \rangle_\lambda = v^2$, $\langle X_2, X_2 \rangle_\lambda = h_\theta^2$, $\langle X_3, X_3 \rangle_\lambda = h_r^2$, and $\langle X_4, X_4 \rangle_\lambda = 1$.
- (2) $[X_i, X_4] = 0$ since each X_i is invariant under the flow of $\frac{\partial}{\partial r}$.
- (3) $\langle [X_i, X_j], X_4 \rangle_\lambda = 0$ since $[X_i, X_j]$ is tangent to level surfaces of r .

By (3) above, there exist constants (a_i) , (b_j) , and (c_k) , $1 \leq i, j, k \leq 3$, such that

$$(3.2) \quad [X_1, X_2] = a_1 X_1 + a_2 X_2 + a_3 X_3$$

$$(3.3) \quad [X_1, X_3] = b_1 X_1 + b_2 X_2 + b_3 X_3$$

$$(3.4) \quad [X_2, X_3] = c_1 X_1 + c_2 X_2 + c_3 X_3.$$

The purpose of Section 5 is to compute these constants.

The orthonormal basis corresponding to (3.1) is

$$(3.5) \quad Y_1 = \frac{1}{v} X_1, \quad Y_2 = \frac{1}{h_\theta} X_2, \quad Y_3 = \frac{1}{h_r} X_3, \quad Y_4 = X_4.$$

Direct calculations show that the Lie brackets for this basis have the following properties:

- (1) $[Y_1, Y_2] = \frac{1}{vh_\theta}[X_1, X_2]$, $[Y_1, Y_3] = \frac{1}{vh_r}[X_1, X_3]$, and $[Y_2, Y_3] = \frac{1}{h_\theta h_r}[X_2, X_3]$.
- (2) $[Y_1, Y_4] = \frac{v}{v}Y_1$, $[Y_2, Y_4] = \frac{h'_\theta}{h_\theta}Y_2$, and $[Y_3, Y_4] = \frac{h'_r}{h_r}Y_3$.

4. COMPONENTS OF THE CURVATURE TENSOR IN \mathbb{CH}^2

The components of the $(4,0)$ curvature tensor of the complex hyperbolic metric g can be expressed in terms of g and the complex structure J . The following formula can be found in [KN96] or in Section 5 of [Bel11] (recall Remark 1.4 from the Introduction). In this formula X, Y, Z , and W are arbitrary vector fields.

$$\begin{aligned} 4\langle R_g(X, Y)Z, W \rangle_g &= \langle X, W \rangle_g \langle Y, Z \rangle_g - \langle X, Z \rangle_g \langle Y, W \rangle_g \\ &\quad + \langle X, JW \rangle_g \langle Y, JZ \rangle_g - \langle X, JZ \rangle_g \langle Y, JW \rangle_g + 2\langle X, JY \rangle_g \langle W, JZ \rangle_g. \end{aligned}$$

Recall that the complex structure J on \mathbb{CH}^2 preserves the complex hyperbolic metric g . That is, for any vector fields X and Y , $\langle X, Y \rangle_g = \langle JX, JY \rangle_g$. We therefore have that

$$JY_1 = Y_2, \quad JY_2 = -Y_1, \quad JY_3 = -Y_4, \quad JY_4 = Y_3.$$

Up to symmetries of the curvature tensor, the following are the only non-zero components of the curvature tensor of the complex hyperbolic metric g with respect to the orthonormal basis (Y_1, Y_2, Y_3, Y_4) .

$$\begin{aligned} (4.1) \quad \langle R_g(Y_1, Y_2)Y_1, Y_2 \rangle_g &= -1 & (4.5) \quad \langle R_g(Y_1, Y_3)Y_2, Y_4 \rangle_g &= \frac{1}{4} \\ (4.2) \quad \langle R_g(Y_1, Y_3)Y_1, Y_3 \rangle_g &= -\frac{1}{4} & (4.6) \quad \langle R_g(Y_1, Y_4)Y_2, Y_3 \rangle_g &= -\frac{1}{4} \\ (4.3) \quad \langle R_g(Y_2, Y_3)Y_2, Y_3 \rangle_g &= -\frac{1}{4} & (4.7) \quad \langle R_g(Y_1, Y_4)Y_1, Y_4 \rangle_g &= -\frac{1}{4} \\ & & (4.8) \quad \langle R_g(Y_2, Y_4)Y_2, Y_4 \rangle_g &= -\frac{1}{4} \\ (4.4) \quad \langle R_g(Y_1, Y_2)Y_3, Y_4 \rangle_g &= \frac{1}{2} & (4.9) \quad \langle R_g(Y_3, Y_4)Y_3, Y_4 \rangle_g &= -1 \end{aligned}$$

Equations (4.1) through (4.9) will be used in the following Section to determine the values of the coefficients in equations (3.2) through (3.4).

5. LIE BRACKETS (PART II)

The purpose of this Section is to compute the coefficients in equations (3.2), (3.3), and (3.4). The major tool is a formula worked out by Belegardek in [Bel12] and stated in Appendix B of [Bel11]. The set up for this formula is as follows. Suppose that $\lambda = dr^2 + \lambda_r$ is a warped product metric on $I \times F$, and let $\{X_i\}$ be a λ_r -orthogonal basis of vector fields on a neighborhood $U \subset F$. Let $h_i(r) = \sqrt{\langle X_i, X_i \rangle_{\lambda_r}}$ so that the collection $Y_i = \frac{1}{h_i}X_i$ forms an orthonormal basis on U for

any $r > 0$. Then (recall Remark 1.4 for the difference between this and [Bel11])

$$(5.1) \quad 2\langle R\left(\frac{\partial}{\partial r}, Y_i\right) Y_j, Y_k \rangle_\lambda = \\ \langle [Y_i, Y_k], Y_j \rangle_\lambda \left(\ln \frac{h_j}{h_k}\right)' + \langle [Y_j, Y_i], Y_k \rangle_\lambda \left(\ln \frac{h_k}{h_j}\right)' + \langle [Y_j, Y_k], Y_i \rangle_\lambda \left(\ln \frac{h_i^2}{h_j h_k}\right)'.$$

As a first (easy) step, we use equation (5.1) to prove the following.

Lemma 5.1. *In equations (3.2), (3.3), and (3.4)*

$$(5.2) \quad a_1 = a_2 = b_1 = b_3 = c_2 = c_3 = 0.$$

Proof. We will prove that $a_1 = 0$, and then indicate how to analogously show that each of the other coefficients in equation (5.2) are zero.

With respect to the complex hyperbolic metric g , we know that $\langle R_g(Y_4, Y_1)Y_1, Y_2 \rangle_g = 0$. But then by equation (5.1) we have that

$$\begin{aligned} 0 &= \langle [Y_1, Y_2], Y_1 \rangle_g \left(\ln \frac{h_v}{h_\theta}\right)' + 0 + \langle [Y_1, Y_2], Y_1 \rangle_g \left(\ln \frac{h_v}{h_\theta}\right)' \\ &= \frac{2}{\sinh^2\left(\frac{r}{2}\right) \cosh\left(\frac{r}{2}\right)} \langle [X_1, X_2], X_1 \rangle_g \left(\ln \left(\frac{\sinh\left(\frac{r}{2}\right)}{\cosh\left(\frac{r}{2}\right)}\right)\right)' \\ &= \frac{2a_1}{\cosh\left(\frac{r}{2}\right)} \left(\ln \tanh\left(\frac{r}{2}\right)\right)' \end{aligned}$$

and thus $a_1 = 0$.

Then to show:

$$\begin{aligned} a_2 &= 0 \text{ use } \langle R_g(Y_4, Y_2)Y_1, Y_2 \rangle_g = 0 \\ b_1 &= 0 \text{ use } \langle R_g(Y_4, Y_1)Y_1, Y_3 \rangle_g = 0 \\ b_3 &= 0 \text{ use } \langle R_g(Y_4, Y_3)Y_1, Y_3 \rangle_g = 0 \\ c_2 &= 0 \text{ use } \langle R_g(Y_4, Y_2)Y_2, Y_3 \rangle_g = 0 \\ c_3 &= 0 \text{ use } \langle R_g(Y_4, Y_3)Y_2, Y_3 \rangle_g = 0. \end{aligned}$$

□

Due to Lemma 5.1, we can rewrite equations (3.2), (3.3), and (3.4) as

$$(5.3) \quad [X_1, X_2] = \alpha_3 X_3$$

$$(5.4) \quad [X_1, X_3] = \alpha_2 X_2$$

$$(5.5) \quad [X_2, X_3] = \alpha_1 X_1$$

where $\alpha_3 = a_3$, $\alpha_2 = b_2$, and $\alpha_1 = c_1$. The reason that the α_i are not zero is because of equations (4.4), (4.5), and (4.6). If these “mixed components” of the curvature tensor for g were zero, then these coefficients would also be zero. In particular, this would be the case if we were dealing with \mathbb{H}^4 instead of \mathbb{CH}^2 .

But we can combine equations (4.4) through (4.6) with (5.1), and use the *Nijenhuis tensor*, to prove the following Theorem.

Theorem 5.2. *The coefficients in equations (5.3), (5.4), and (5.5) are*

$$(5.6) \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = -\frac{1}{2}.$$

Proof. First, note that

$$(5.7) \quad [Y_1, Y_2] = \frac{\alpha_3 h_r}{v h_\theta} Y_3, \quad [Y_1, Y_3] = \frac{\alpha_2 h_\theta}{v h_r} Y_2, \quad [Y_2, Y_3] = \frac{\alpha_1 v}{h_\theta h_r} Y_1$$

where, since we are dealing with the complex hyperbolic metric g , $v = \sinh\left(\frac{r}{2}\right)$, $h_\theta = \cosh\left(\frac{r}{2}\right)$, and $h_r = \cosh(r)$.

We first combine equations (4.5) and (4.6) with equation (5.1) to obtain

$$(5.8) \quad \begin{aligned} -\frac{1}{2} &= \langle [Y_2, Y_3], Y_1 \rangle_g \left(\ln \frac{v}{h_r} \right)' + \langle [Y_1, Y_2], Y_3 \rangle_g \left(\ln \frac{h_r}{v} \right)' + \langle [Y_1, Y_3], Y_2 \rangle_g \left(\ln \frac{h_\theta^2}{v h_r} \right)' \\ &= \frac{a_1 v}{h_\theta h_r} \left(\ln \frac{v}{h_r} \right)' - \frac{a_3 h_r}{v h_\theta} \left(\ln \frac{v}{h_r} \right)' + \frac{a_2 h_\theta}{v h_r} \left(\ln \frac{h_\theta^2}{v h_r} \right)' \\ &= \left(\frac{\alpha_1 \sinh\left(\frac{r}{2}\right)}{\cosh\left(\frac{r}{2}\right) \cosh(r)} - \frac{\alpha_3 \cosh(r)}{\sinh\left(\frac{r}{2}\right) \cosh\left(\frac{r}{2}\right)} \right) \left(\frac{1}{2} \coth\left(\frac{r}{2}\right) - \tanh(r) \right) \\ &\quad + \frac{\alpha_2 \cosh\left(\frac{r}{2}\right)}{\sinh\left(\frac{r}{2}\right) \cosh(r)} \left(\tanh\left(\frac{r}{2}\right) - \frac{1}{2} \coth\left(\frac{r}{2}\right) - \tanh(r) \right) \end{aligned}$$

and

$$(5.9) \quad \begin{aligned} -\frac{1}{2} &= \langle [Y_1, Y_2], Y_3 \rangle_g \left(\ln \frac{h_r}{h_\theta} \right)' + \langle [Y_3, Y_1], Y_2 \rangle_g \left(\ln \frac{h_\theta}{h_r} \right)' + \langle [Y_3, Y_2], Y_1 \rangle_g \left(\ln \frac{v^2}{h_\theta h_r} \right)' \\ &= \frac{a_3 h_r}{v h_\theta} \left(\ln \frac{h_r}{h_\theta} \right)' + \frac{a_2 h_\theta}{v h_r} \left(\ln \frac{h_r}{h_\theta} \right)' - \frac{a_1 v}{h_\theta h_r} \left(\ln \frac{v^2}{h_\theta h_r} \right)' \\ &= \left(\frac{\alpha_3 \cosh(r)}{\sinh\left(\frac{r}{2}\right) \cosh\left(\frac{r}{2}\right)} + \frac{\alpha_2 \cosh\left(\frac{r}{2}\right)}{\sinh\left(\frac{r}{2}\right) \cosh(r)} \right) \left(\tanh(r) - \frac{1}{2} \tanh\left(\frac{r}{2}\right) \right) \\ &\quad - \frac{\alpha_1 \sinh\left(\frac{r}{2}\right)}{\cosh\left(\frac{r}{2}\right) \cosh(r)} \left(\coth\left(\frac{r}{2}\right) - \frac{1}{2} \tanh\left(\frac{r}{2}\right) - \tanh(r) \right). \end{aligned}$$

This yields two equations with three unknown variables. It is a tedious exercise to check that the values in Theorem 5.2 satisfy these two equations. But, of course, there is no reason to believe (yet) that this solution is unique. In an attempt to obtain a third independent equation we could combine equation (4.4) with equation (5.1), but one easily checks that this leads to a dependent system of equations.

There are two routes to obtaining a third independent equation involving the α'_i s. One way is to compute the components of R_λ as equations involving the α'_i s and v , h_θ , and h_r . We could then substitute in the values for g to obtain several other independent equations. But the easier method is to use the *Nijenhuis Tensor*. Since the almost complex structure on \mathbb{CH}^2 is integrable, we have that

$$(5.10) \quad 0 = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for all vector fields X and Y . Evaluating (5.10) at $X = Y_1$ and $Y = Y_3$ yields

$$\begin{aligned}
0 &= [Y_1, Y_3] + J[Y_2, Y_3] + J[Y_1, -Y_4] - [Y_2, -Y_4] \\
&= \frac{\alpha_2 h_\theta}{v h_r} Y_2 + \frac{\alpha_1 v}{h_\theta h_r} JY_1 - \frac{v'}{v} JY_1 + \frac{h'_\theta}{h_\theta} Y_2 \\
&= \left(\frac{\alpha_2 \cosh\left(\frac{r}{2}\right)}{\sinh\left(\frac{r}{2}\right) \cosh(r)} + \frac{\alpha_1 \cosh(r)}{\cosh\left(\frac{r}{2}\right) \sinh\left(\frac{r}{2}\right)} - \frac{1}{2} \coth\left(\frac{r}{2}\right) + \frac{1}{2} \tanh\left(\frac{r}{2}\right) \right) Y_2 \\
(5.11) \implies & \frac{1}{\cosh(r)} \left(\coth\left(\frac{r}{2}\right) \alpha_2 + \tanh\left(\frac{r}{2}\right) \alpha_1 \right) = \frac{1}{\sinh(r)}.
\end{aligned}$$

Now one easily checks that equations (5.8), (5.9), and (5.11) are independent, guaranteeing a unique solution. It is also easy this time to check that the values in Theorem 5.2 satisfy equation (5.11), completing the proof of Theorem 5.2. \square

The following lemma just restates what has been proven so far:

Lemma 5.3. *The six independent Lie brackets for the vector fields described in equation (3.5) are*

$$\begin{aligned}
[Y_1, Y_2] &= \frac{-h_r}{2v h_\theta} Y_3, & [Y_1, Y_3] &= \frac{h_\theta}{2v h_r} Y_2, & [Y_2, Y_3] &= \frac{v}{2h_\theta h_r} Y_1 \\
[Y_1, Y_4] &= \frac{v'}{v} Y_1, & [Y_2, Y_4] &= \frac{h'_\theta}{h_\theta} Y_2, & [Y_3, Y_4] &= \frac{h'_r}{h_r} Y_3.
\end{aligned}$$

6. THE LEVI-CIVITA CONNECTION OF λ

In this Subsection we will compute the Levi-Civita connection ∇ associated to the metric λ with respect to the frame (Y_1, Y_2, Y_3, Y_4) . To perform this calculation we will use the well-known ‘‘Koszul formula’’ (which can be found on pg. 55 of [doC92])

$$\begin{aligned}
(6.1) \quad \langle \nabla_Y X, Z \rangle_\lambda &= \frac{1}{2} (X \langle Y, Z \rangle_\lambda + Y \langle Z, X \rangle_\lambda - Z \langle X, Y \rangle_\lambda \\
&\quad - \langle [X, Z], Y \rangle_\lambda - \langle [Y, Z], X \rangle_\lambda + \langle [X, Y], Z \rangle_\lambda).
\end{aligned}$$

Since we are considering an orthonormal frame we know that $\langle Y_i, Y_j \rangle_\lambda = \delta_{ij}$, where δ_{ij} denotes Kronecker’s delta. Therefore the first three terms on the right hand side of formula (6.1) are all zero in our given frame. Thus, formula (6.1) reduces to

$$(6.2) \quad \langle \nabla_Y X, Z \rangle_\lambda = -\frac{1}{2} (\langle [X, Z], Y \rangle_\lambda + \langle [Y, Z], X \rangle_\lambda + \langle [X, Y], Z \rangle_\lambda).$$

It is now a simple calculation using formula (6.2) and the results of Lemma 5.3 to prove the following Theorem.

Theorem 6.1. *The Levi-Civita connection ∇ is given by the 16 equations*

$$(6.3) \quad 0 = \nabla_{Y_4} Y_1 = \nabla_{Y_4} Y_2 = \nabla_{Y_4} Y_3 = \nabla_{Y_4} Y_4$$

$$(6.4) \quad \nabla_{Y_1} Y_1 = -\frac{v'}{v} Y_4, \quad \nabla_{Y_2} Y_2 = -\frac{h'_\theta}{h_\theta} Y_4, \quad \nabla_{Y_3} Y_3 = -\frac{h'_r}{h_r} Y_4$$

$$(6.5) \quad \nabla_{Y_1} Y_4 = \frac{v'}{v} Y_1, \quad \nabla_{Y_2} Y_4 = \frac{h'_\theta}{h_\theta} Y_2, \quad \nabla_{Y_3} Y_4 = \frac{h'_r}{h_r} Y_3$$

$$(6.6) \quad \nabla_{Y_1} Y_2 = -\frac{1}{2} \left(\frac{v}{2h_\theta h_r} + \frac{h_\theta}{2vh_r} + \frac{h_r}{2vh_\theta} \right) Y_3$$

$$(6.7) \quad \nabla_{Y_1} Y_3 = -\frac{1}{2} \left(\frac{-v}{2h_\theta h_r} - \frac{h_\theta}{2vh_r} - \frac{h_r}{2vh_\theta} \right) Y_2$$

$$(6.8) \quad \nabla_{Y_2} Y_1 = -\frac{1}{2} \left(\frac{v}{2h_\theta h_r} + \frac{h_\theta}{2vh_r} - \frac{h_r}{2vh_\theta} \right) Y_3$$

$$(6.9) \quad \nabla_{Y_2} Y_3 = -\frac{1}{2} \left(\frac{-v}{2h_\theta h_r} - \frac{h_\theta}{2vh_r} + \frac{h_r}{2vh_\theta} \right) Y_1$$

$$(6.10) \quad \nabla_{Y_3} Y_1 = -\frac{1}{2} \left(\frac{-v}{2h_\theta h_r} + \frac{h_\theta}{2vh_r} - \frac{h_r}{2vh_\theta} \right) Y_2$$

$$(6.11) \quad \nabla_{Y_3} Y_2 = -\frac{1}{2} \left(\frac{v}{2h_\theta h_r} - \frac{h_\theta}{2vh_r} + \frac{h_r}{2vh_\theta} \right) Y_1$$

Proof. We only prove equation (6.6) and the first equalities in equations (6.3), (6.4), and (6.5). All of the other equations are proven analogously. As stated above, each equation is obtained by simply combining Lemma 5.3 with formula (6.2). To prove the first equality in (6.3) we compute

$$\langle \nabla_{Y_4} Y_1, Y_i \rangle_\lambda = -\frac{1}{2} (\langle [Y_1, Y_i], Y_4 \rangle_\lambda + \langle [Y_4, Y_i], Y_1 \rangle_\lambda + \langle [Y_1, Y_4], Y_i \rangle_\lambda).$$

Using Lemma 5.3 we see that each term above is zero when $i = 2, 3$, or 4 . When $i = 1$, the first term is zero and the last two terms cancel due to the antisymmetry of the Lie bracket. Thus, $\nabla_{Y_4} Y_1 = 0$.

To prove the first equality in (6.4) we plug into formula (6.2) to obtain

$$\begin{aligned} \langle \nabla_{Y_1} Y_1, Y_i \rangle_\lambda &= -\frac{1}{2} (\langle [Y_1, Y_i], Y_1 \rangle_\lambda + \langle [Y_1, Y_i], Y_1 \rangle_\lambda + \langle [Y_1, Y_1], Y_i \rangle_\lambda) \\ &= -\langle [Y_1, Y_i], Y_1 \rangle_\lambda. \end{aligned}$$

This is nonzero only when $i = 4$, and substituting the value of $[Y_1, Y_4]$ given in Lemma 5.3 yields the desired result.

For the first equality in (6.5) we have

$$\langle \nabla_{Y_1} Y_4, Y_i \rangle_\lambda = -\frac{1}{2} (\langle [Y_4, Y_i], Y_1 \rangle_\lambda + \langle [Y_1, Y_i], Y_4 \rangle_\lambda + \langle [Y_4, Y_1], Y_i \rangle_\lambda).$$

Each term is zero when $i = 2, 3$ or 4 . When $i = 1$ we have

$$\langle \nabla_{Y_1} Y_4, Y_1 \rangle_\lambda = -\langle [Y_4, Y_1], Y_1 \rangle_\lambda = -\left(-\frac{v'}{v} \langle Y_1, Y_1 \rangle_\lambda \right) = \frac{v'}{v}.$$

Lastly, to verify equation (6.6), we have from formula (6.2)

$$\langle \nabla_{Y_1} Y_2, Y_i \rangle_\lambda = -\frac{1}{2} (\langle [Y_2, Y_i], Y_1 \rangle_\lambda + \langle [Y_1, Y_i], Y_2 \rangle_\lambda + \langle [Y_2, Y_1], Y_i \rangle_\lambda).$$

We see from Lemma 5.3 that each term above is zero when $i = 1, 2$, or 4 . When $i = 3$ we have that

$$\begin{aligned}\langle \nabla_{Y_1} Y_2, Y_3 \rangle_\lambda &= -\frac{1}{2} (\langle [Y_2, Y_3], Y_1 \rangle_\lambda + \langle [Y_1, Y_3], Y_2 \rangle_\lambda - \langle [Y_1, Y_2], Y_3 \rangle_\lambda) \\ &= -\frac{1}{2} \left(\frac{v}{2h_\theta h_r} + \frac{h_\theta}{2vh_r} - \frac{-h_r}{2vh_\theta} \right).\end{aligned}$$

□

7. COMPONENTS OF THE CURVATURE TENSOR OF λ

Recall from Remark 1.4 that in this paper we are using the definition

$$(7.1) \quad R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

for the curvature tensor. Let R_λ denote the curvature tensor of the metric λ . Then, up to symmetries of the curvature tensor, the only non-zero components of R_λ are given by the following Theorem.

Theorem 7.1. *In terms of the basis given in equation (3.5), the only independent nonzero components of the $(4, 0)$ curvature tensor R_λ are the following:*

$$\begin{aligned}\langle R_\lambda(Y_1, Y_2)Y_1, Y_2 \rangle_\lambda &= -\frac{v'h'_\theta}{vh_\theta} - \frac{1}{4} \left(\frac{-v^2}{4h_\theta^2 h_r^2} - \frac{h_\theta^2}{4v^2 h_r^2} + \frac{3h_r^2}{4v^2 h_\theta^2} - \frac{1}{2v^2} + \frac{1}{2h_\theta^2} - \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_1, Y_3)Y_1, Y_3 \rangle_\lambda &= -\frac{v'h'_r}{vh_r} - \frac{1}{4} \left(\frac{-v^2}{4h_\theta^2 h_r^2} + \frac{3h_\theta^2}{4v^2 h_r^2} - \frac{h_r^2}{4v^2 h_\theta^2} - \frac{1}{2v^2} - \frac{1}{2h_\theta^2} + \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_2, Y_3)Y_2, Y_3 \rangle_\lambda &= -\frac{h'_\theta h'_r}{h_\theta h_r} - \frac{1}{4} \left(\frac{3v^2}{4h_\theta^2 h_r^2} - \frac{h_\theta^2}{4v^2 h_r^2} - \frac{h_r^2}{4v^2 h_\theta^2} + \frac{1}{2v^2} + \frac{1}{2h_\theta^2} + \frac{1}{2h_r^2} \right) \\ \langle R_\lambda(Y_1, Y_2)Y_3, Y_4 \rangle_\lambda &= \frac{-1}{4h_r} \left[\left(\frac{h_\theta}{v} \right)' - \left(\frac{v}{h_\theta} \right)' - \left(\frac{h_r^2}{vh_\theta} \right)' \right] \\ \langle R_\lambda(Y_1, Y_3)Y_2, Y_4 \rangle_\lambda &= \frac{-1}{4h_\theta} \left[-\left(\frac{h_r}{v} \right)' + \left(\frac{v}{h_r} \right)' + \left(\frac{h_\theta^2}{vh_r} \right)' \right] \\ \langle R_\lambda(Y_1, Y_4)Y_2, Y_3 \rangle_\lambda &= \frac{-1}{4v} \left[\left(\frac{h_\theta}{h_r} \right)' + \left(\frac{h_r}{h_\theta} \right)' + \left(\frac{v^2}{h_\theta h_r} \right)' \right] \\ \langle R_\lambda(Y_1, Y_4)Y_1, Y_4 \rangle_\lambda &= -\frac{v''}{v} & \langle R_\lambda(Y_2, Y_4)Y_2, Y_4 \rangle_\lambda &= -\frac{h''_\theta}{h_\theta} \\ \langle R_\lambda(Y_3, Y_4)Y_3, Y_4 \rangle_\lambda &= -\frac{h''_r}{h_r}.\end{aligned}$$

It is an exercise in hyperbolic trigonometric identities to check that, when $v = \sinh(\frac{r}{2})$, $h_\theta = \cosh(\frac{r}{2})$, and $h_r = \cosh(r)$, the above formulas reduce to the constants in equations (4.1) through (4.9). Also, note that the curvature tensor on a four dimensional Riemannian manifold has 21 components which are independent with respect to the symmetries of the curvature tensor. So Theorem 7.1 also states that the remaining 12 components of R_λ are identically zero.

Proof. There is nothing enlightening about computing the 21 independent components of R_λ . We simply plug the results of Lemma 5.3 and Theorem 6.1 into equation (7.1) and punch out the computations using the properties of ∇ (and remembering that $Y_4 = \frac{\partial}{\partial r}$ and all of v , h_θ , and h_r are functions of r). As an illustration, let us prove the first equality in Theorem 7.1. In what follows, $a = v/(2h_\theta h_r)$, $b = h_\theta/(2vh_r)$, and $c = h_r/(2vh_\theta)$.

$$\begin{aligned}
 \langle R_\lambda(Y_1, Y_2)Y_1, Y_2 \rangle_\lambda &= \langle \nabla_{Y_2} \nabla_{Y_1} Y_1 - \nabla_{Y_1} \nabla_{Y_2} Y_1 + \nabla_{[Y_1, Y_2]} Y_1, Y_2 \rangle_\lambda \\
 &= \left\langle -\frac{v'}{v} \nabla_{Y_2} Y_4 + \frac{1}{2}(a + b - c) \nabla_{Y_1} Y_3 - c \nabla_{Y_3} Y_1, Y_2 \right\rangle_\lambda \\
 &= \left\langle -\frac{v' h'_\theta}{v h_\theta} Y_2 - \frac{1}{4}(a + b - c)(-a - b - c) Y_2 + \frac{1}{2}c(-a + b - c) Y_2, Y_2 \right\rangle_\lambda \\
 &= -\frac{v' h'_\theta}{v h_\theta} - \frac{1}{4}(-a^2 - b^2 + 3c^2 - 2ab + 2ac - 2bc) \\
 &= -\frac{v' h'_\theta}{v h_\theta} - \frac{1}{4} \left(\frac{-v^2}{4h_\theta^2 h_r^2} - \frac{h_\theta^2}{4v^2 h_r^2} + \frac{3h_r^2}{4v^2 h_\theta^2} - \frac{1}{2v^2} + \frac{1}{2h_\theta^2} - \frac{1}{2h_r^2} \right).
 \end{aligned}$$

□

8. THE CASE WHEN $h_\theta = h_r$

In Sections 10 and 11 we will construction functions for v , h_θ , and h_r for which λ will be complete, have finite volume, and have negative curvature bounded away from zero. The purpose of Sections 8 and 9 are to derive formulas and results which will help to prove that the metric constructed in Section 10 has sectional curvature bounded above by a negative constant. The formulas in Theorem 7.1 are rather long and complicated, but in the very special case when $h_\theta = h_r := h$ these formulas reduce very nicely as stated in the following Corollary. In what follows, $K(Y_i, Y_j) := \langle R(Y_i, Y_j)Y_i, Y_j \rangle_\lambda$ denotes the sectional curvature (with respect to λ) of the 2-plane spanned by (Y_i, Y_j) .

Corollary 8.1 (Corollary to Theorem 7.1). *When $h_\theta = h_r := h$, the formulas in Theorem 7.1 reduce to*

$$\begin{aligned}
 K(Y_1, Y_2) &= K(Y_1, Y_3) = -\frac{v' h'}{v h} + \frac{v^2}{16h^4} \\
 K(Y_2, Y_3) &= -\frac{1}{4h^2} - \frac{3v^2}{16h^4} - \left(\frac{h'}{h} \right)^2 \\
 K(Y_1, Y_4) &= -\frac{v''}{v} & K(Y_2, Y_4) &= K(Y_3, Y_4) = -\frac{h''}{h} \\
 \langle R_\lambda(Y_1, Y_4)Y_2, Y_3 \rangle_\lambda &= \frac{-1}{4v} \left(\frac{v^2}{h^2} \right)' = \frac{-v}{2h^2} \left(\ln \frac{v}{h} \right)' \\
 \langle R_\lambda(Y_1, Y_2)Y_3, Y_4 \rangle_\lambda &= \frac{v}{4h^2} \left(\ln \frac{v}{h} \right)' = -\langle R_\lambda(Y_1, Y_3)Y_2, Y_4 \rangle_\lambda
 \end{aligned}$$

Remark 8.2. It is important to note that the complex hyperbolic metric g on $\mathbb{CH}^2 \setminus \mathbb{H}^2$ does *not* satisfy the conditions of Corollary 8.1. Namely, $h_\theta = \cosh\left(\frac{r}{2}\right) \neq$

$\cosh(r) = h_r$. But we will use the above formulas in Sections 10 and 11 in order to greatly reduce calculations.

Comparing the equations in Corollary 8.1 with equations (9.2) through (9.5) in [Bel11], one sees that they are nearly identical. The only difference is that the above equations contain specific (and inconsistent) values for the constant c_{23} in [Bel11]. More specifically, the constant c_{23} takes on the values of 0 and $1/2$ in the equation for $K(Y_2, Y_3)$ (from left to right), and takes on the value of $1/2$ in the formula for $\langle R_\lambda(Y_1, Y_2)Y_3, Y_4 \rangle_\lambda$.

In order to simplify calculations even further, let us prove the following Lemma before computing a formula for the sectional curvature of a generic 2-plane when $h_\theta = h_r$.

Lemma 8.3. *Assume that $h_\theta = h_r := h$, and suppose that (U_2, U_3) is a orthonormal set of vectors (with respect to λ) whose span is the plane spanned by (Y_2, Y_3) . Furthermore, assume that (U_2, U_3) and (Y_2, Y_3) have the same orientation. Then the curvature formulas in Corollary 8.1 remain unchanged if Y_i is replaced with U_i for $i = 2, 3$.*

Proof. By the assumptions, there exists constants a_2, a_3, b_2, b_3 such that

$$\begin{aligned} U_2 &= a_2 Y_2 + a_3 Y_3, & U_3 &= b_2 Y_2 + b_3 Y_3 \\ a_2^2 + a_3^2 &= 1 = b_2^2 + b_3^2 \\ a_2 b_2 + a_3 b_3 &= 0 \end{aligned}$$

Now notice that

$$\begin{aligned} (a_2 b_3 - a_3 b_2)^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_2 b_2 + a_3 b_3)^2 \\ &= (a_2^2 + a_3^2)(b_2^2 + b_3^2) \\ &= 1 \end{aligned}$$

and therefore $a_2 b_3 - a_3 b_2 = \pm 1$. But since (U_2, U_3) and (Y_2, Y_3) have the same orientation, we have that $a_2 b_3 - a_3 b_2 = 1$.

Now we just use this formula in conjunction with the formulas in Corollary 8.1:

$$\begin{aligned} K(Y_1, U_2) &= \langle R(Y_1, U_2)Y_1, U_2 \rangle = a_2^2 \langle R(Y_1, Y_2)Y_1, Y_2 \rangle + a_3^2 \langle R(Y_1, Y_3)Y_1, Y_3 \rangle \\ &= (a_2^2 + a_3^2) \left(-\frac{v'h'}{vh} + \frac{v^2}{16h^4} \right) = -\frac{v'h'}{vh} + \frac{v^2}{16h^4}. \end{aligned}$$

The proof that $K(Y_1, U_3) = K(Y_1, Y_3)$ is completely analogous, as is the proof that $K(U_2, Y_4) = K(U_3, Y_4) = K(Y_2, Y_4) = K(Y_3, Y_4)$. Also, it is clear that $K(U_2, U_3) = K(Y_2, Y_3)$. So all that is left is to check the ‘‘cross-terms’’ of the curvature tensor. Each of these three cases are nearly identical, so we only compute one here.

$$\begin{aligned} \langle R(Y_1, U_2)U_3, Y_4 \rangle &= \langle R(Y_1, a_2 Y_2 + a_3 Y_3)b_2 Y_2 + b_3 Y_3, Y_4 \rangle \\ &= a_2 b_3 \langle R(Y_1, Y_2)Y_3, Y_4 \rangle + a_3 b_2 \langle R(Y_1, Y_3)Y_2, Y_4 \rangle \\ &= (a_2 b_3 - a_3 b_2) \langle R(Y_1, Y_2)Y_3, Y_4 \rangle \\ &= \langle R(Y_1, Y_2)Y_3, Y_4 \rangle \end{aligned}$$

and where the third equality above is due to the specific formulas in Corollary 8.1. \square

Using the notation of Section 2, let $p \in (0, \infty) \times F$ and let $q = \pi(p) \in \mathbb{H}^2$. Let σ denote a generic 2-plane tangent to $(0, \infty) \times F$ at p . In the generic case when $d\pi(\sigma) = \mathbb{H}^2$, an identical argument to that given by Belegradek in ([Bel11], Section 9, pg. 559) shows that there exists an orthonormal basis (C, D) of σ such that

$$C = c_1 Y_1 + c_2 U_2 + c_3 U_3 + c_4 Y_4, \quad D = d_1 Y_1 + d_2 U_2$$

and (U_2, U_3) are an orthonormal set as in Lemma 8.3.

In the following calculation we use some new notation. Since we proved in Lemma 8.3 the curvature formulas are identical, in what follows we replace (U_2, U_3) with (Y_2, Y_3) , respectively. Also, we use the convention $R_{ijkl} := \langle R(Y_i, Y_j)Y_k, Y_l \rangle_\lambda$.

We now compute:

$$\begin{aligned} K(\sigma) &= K(C, D) = d_1^2 K(C, Y_1) + d_2^2 K(C, Y_2) + 2d_1 d_2 \langle R(C, Y_1)C, Y_2 \rangle \\ K(C, Y_1) &= c_2^2 K(Y_1, Y_2) + c_3^2 K(Y_1, Y_3) + c_4^2 K(Y_1, Y_4) \\ K(C, Y_2) &= c_1^2 K(Y_1, Y_2) + c_3^2 K(Y_2, Y_3) + c_4^2 K(Y_2, Y_4). \end{aligned}$$

Also,

$$\begin{aligned} \langle R(C, Y_1)C, Y_2 \rangle &= -c_1 c_2 K(Y_1, Y_2) + c_3 c_4 R_{1324} + c_3 c_4 R_{1423} \\ &= -c_1 c_2 K(Y_1, Y_2) + \frac{3}{2} c_3 c_4 R_{1423} \end{aligned}$$

since $R_{1324} = \frac{1}{2} R_{1423}$ whenever $h_\theta = h_r$.

Putting this all together gives that

$$\begin{aligned} (8.1) \quad K(\sigma) &= (c_1 d_2 - c_2 d_1)^2 K(Y_1, Y_2) + d_1^2 c_3^2 K(Y_1, Y_3) + d_1^2 c_4^2 K(Y_1, Y_4) \\ &\quad + d_2^2 c_3^2 K(Y_2, Y_3) + d_2^2 c_4^2 K(Y_2, Y_4) + 3c_3 c_4 d_1 d_2 R_{1423}. \end{aligned}$$

In an identical manner as Remark 9.6 in [Bel11], if $R_{1423} = 0$ and $K(Y_i, Y_j)$ is less than or equal to a negative constant for each $i \neq j$ then $K(\sigma)$ is bounded above by the same negative constant. This is because the coefficients of the sectional curvatures of the coordinate planes in equation (8.1) add up to one.

Lastly, the case when $d\pi(\sigma) \neq \mathbb{H}^2$ is identical to Remark 9.7 in [Bel11]. If $\dim(d\pi(\sigma)) = 0$, then $K(\sigma) = K(Y_1, Y_4)$. If $\dim(d\pi(\sigma)) = 1$, then we can choose our orthonormal basis (C, D) in such a way that $K(\sigma)$ contains no “mixed terms” (such as R_{1423} in equation (8.1)). Therefore in the case when $\dim(d\pi(\sigma)) \neq 2$, if the sectional curvatures of the coordinate planes are bounded above by a negative constant, so is $K(\sigma)$.

9. SECTIONAL CURVATURE OF λ

The purpose of this section is to set-up and prove Lemma 9.1, Lemma 9.4, and Corollary 9.5 below, which will help us deal with the general case when $h_\theta \neq h_r$. As above, let $p \in (0, \infty) \times F$ and let σ denote a 2-plane tangent to $(0, \infty) \times F$ at p . We can always find an orthonormal basis (A, B) of σ such that

$$A = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4, \quad B = b_2 Y_2 + b_3 Y_3 + b_4 Y_4.$$

Then, using notation from Section 8, we have that

$$\begin{aligned}
 (9.1) \quad K(\sigma) &= \langle R(A, B)A, B \rangle_\lambda \\
 &= a_1^2 b_2^2 R_{1212} + a_1^2 b_3^2 R_{1313} + a_1^2 b_4^2 R_{1414} + (a_2 b_3 - a_3 b_2)^2 R_{2323} + (a_2 b_4 - a_4 b_2)^2 R_{2424} \\
 &\quad + (a_3 b_4 - a_4 b_3)^2 R_{3434} + 2a_1 b_2 (a_3 b_4 - a_4 b_3) R_{1234} \\
 &\quad + 2a_1 b_3 (a_2 b_4 - a_4 b_2) R_{1324} + 2a_1 b_4 (a_2 b_3 - a_3 b_2) R_{1423}.
 \end{aligned}$$

Equation (9.1) is used to prove the following.

Lemma 9.1. *Suppose that $R_{1234}, R_{1324}, R_{1423} \neq 0$, and*

$$\begin{aligned}
 (1a) \quad R_{1212} &\leq -|R_{1234}| & (1b) \quad R_{3434} &\leq -|R_{1234}| \\
 (2a) \quad R_{1313} &\leq -|R_{1324}| & (2b) \quad R_{2424} &\leq -|R_{1324}| \\
 (3a) \quad R_{1414} &\leq -|R_{1423}| & (3b) \quad R_{2323} &\leq -|R_{1423}|.
 \end{aligned}$$

Then there exists $C < 0$ such that $K(\sigma) < C$ for any 2-plane σ .

Remark 9.2. Obviously, Lemma 9.1 also applies in the special case when $h_\theta = h_r$.

Remark 9.3. Also, it is ok if some of R_{1234}, R_{1324} , or $R_{1423} = 0$. But then for the conclusion of the Lemma to be true we need the inequalities using these components with value 0 to be strict.

Proof. With a little bit of arithmetic, one can rewrite equation (9.1) for $K(\sigma)$ as:

$$\begin{aligned}
 &-(\pm a_1 b_2 + a_3 b_4 - a_4 b_3)^2 |R_{1234}| \\
 &\quad + a_1^2 b_2^2 (R_{1212} + |R_{1234}|) + (a_3 b_4 - a_4 b_3)^2 (R_{3434} + |R_{1234}|) \\
 &-(\pm a_1 b_3 + a_2 b_4 - a_4 b_2)^2 |R_{1324}| \\
 &\quad + a_1^2 b_3^2 (R_{1313} + |R_{1324}|) + (a_2 b_4 - a_4 b_2)^2 (R_{2424} + |R_{1324}|) \\
 &-(\pm a_1 b_4 + a_2 b_3 - a_3 b_2)^2 |R_{1423}| \\
 &\quad + a_1^2 b_4^2 (R_{1414} + |R_{1423}|) + (a_2 b_3 - a_3 b_2)^2 (R_{2323} + |R_{1423}|)
 \end{aligned}$$

where the signs of the three “ \pm ” terms depend on the signs of R_{1234} , R_{1324} , and R_{1423} , respectively. For example, if $R_{1234} < 0$ then the first term would be “ $-a_1 b_2$ ”, and similarly for $R_{1234} > 0$ and for the other two terms.

Due to the inequalities in the assumptions of the Lemma, each term above is nonpositive. A tedious but mostly trivial algebraic argument (which makes great use of the fact that (A, B) is orthonormal) can be used to show that the coefficients of the first, fourth, and seventh terms cannot all simultaneously be 0. Thus $K(\sigma) < 0$ for each fixed σ .

Let M denote the maximal value attained by $K()$ over all orthonormal pairs (A, B) . By compactness M is attained via some $K(\sigma)$, and due to the last paragraph this maximum is strictly less than 0. Thus, $K(\sigma) \leq M < 0$ for all 2-planes $\sigma \in T_p M$. \square

The following “Continuity Lemma” is a direct application of the equation in the preceding proof. This Lemma will be used to deal with the endpoints of the regions in Sections 10 and 11, as well as to prove Corollary 9.5.

Lemma 9.4 (Continuity Lemma). *Let λ, λ' be Riemannian metrics on a manifold M , and fix a point $p \in M$. Let R, R' and K, K' denote the curvature tensors and sectional curvatures of λ and λ' , respectively. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|R_{ijkl} - R'_{ijkl}| < \delta$ for all i, j, k, l (at p), then $|K(\sigma) - K'(\sigma)| < \varepsilon$ for all $\sigma \in T_p M$.*

Corollary 9.5. *Suppose that $R_{1234}, R_{1324}, R_{1423} \neq 0$, and that five of the six inequalities assumed in Lemma 9.1 hold. Then there exists $\delta > 0$ such that, if the sixth inequality fails by at most δ , then $K()$ is still bounded above by a negative constant.*

Proof. If the sixth inequality is instead a strict equality, then by Lemma 9.1 $K()$ is bounded above by a negative constant. Then just apply Lemma 9.4 to the two components of the curvature tensor that appear in that sixth inequality. \square

We will make great use of Corollary 9.5 in the final step in Section 11.

10. CONSTRUCTING A COMPLETE NEGATIVELY CURVED METRIC WITH FINITE VOLUME

Our goal is to construct functions v, h_θ , and h_r so that the metric λ is complete, has finite volume, and so that the sectional curvature of λ is bounded above by a negative constant. In this Section we simply construct these functions. In Section 11 we prove the desired properties of the metric. It will be clear that the metric is complete and has finite volume (or see Remark 3.3 of [Bel12]), so we will prove that the sectional curvature is bounded above by a negative constant. The metric λ will also agree with the complex hyperbolic metric g when r is at least half of the normal injectivity radius of the compact totally geodesic submanifold S . In this construction, the domain of v, h_θ , and h_r will be from negative infinity to the normal injectivity radius of S , turning S into a cusp of $M \setminus S$.

There are seven stages to this process, as illustrated in Figure 1. The endpoints of the regions for the stages will all depend on a small positive constant ε and will be defined below.

In the first region, whose domain is $(-\infty, a_\varepsilon)$ where $a_\varepsilon < 0$, we set $v = \varepsilon e^r$ and $h_\theta = h_r = e^{\frac{r}{2}}$. In region 2, whose domain is $(a_\varepsilon, b_\varepsilon)$ with $b_\varepsilon > 0$, we simultaneously “warp” both h_θ and h_r from $e^{\frac{r}{2}}$ to $\cosh(\frac{r}{2})$. Then in region 3, defined on $(b_\varepsilon, c_\varepsilon)$, we simply have that $v = \varepsilon e^r$ and $h_\theta = h_r = \cosh(\frac{r}{2})$. Region 4 occurs over the interval $(c_\varepsilon, d_\varepsilon)$. Here we “bend” v from εe^r to $\sinh(\frac{r}{2})$, while keeping $h_\theta = h_r = \cosh(\frac{r}{2})$. So on region 5, occurring from $(d_\varepsilon, e_\varepsilon)$, we have $v = \sinh(\frac{r}{2})$ and $h_\theta = h_r = \cosh(\frac{r}{2})$.

Notice that in these first five regions we have been very careful to keep $h_\theta = h_r$ so that we can use the much simpler formulas in both Corollary 8.1 and equation (8.1). Since these formulas are very similar to those obtained in [Bel11], much of this work can be copied over in order to compute the curvature bounds in these regions (and our warping functions will essentially agree with Belegradek’s). Region 7, defined from f_ε to the normal injectivity radius of S , is simple since here λ will agree with the complex hyperbolic metric g and thus the curvature will be bounded above by $-\frac{1}{4}$. But in region 6, defined on $(e_\varepsilon, f_\varepsilon)$, we warp h_r from $\cosh(\frac{r}{2})$ to $\cosh(r)$. So in

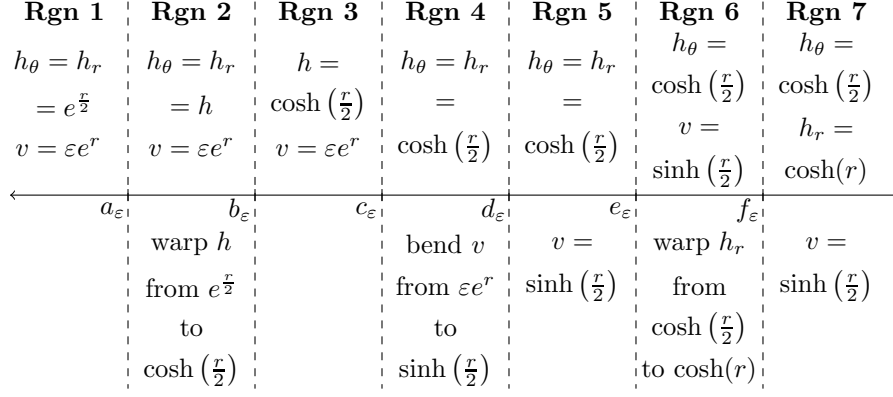


FIGURE 1. Schematic diagram for the warping functions for λ , where “Rgn” stands for “Region”. Note that the diagram is not remotely close to scale. The interval $(a_\varepsilon, b_\varepsilon)$ is very large, while all of the other intervals are arbitrarily small.

this region we have $h_\theta \neq h_r$, forcing us to use the equations in Theorem 7.1. Also, since this is where our case differs from that in [Bel11], we need to come up with a new function to vary from $\cosh\left(\frac{r}{2}\right)$ to $\cosh(r)$. Developing this function (which will just be a cubic polynomial) and proving that we can use Lemma 9.1 and Corollary 9.5 will occupy most of the remainder of this Section and Section 11.

Remark 10.1. Note that none of the regions above contain either of its endpoints. We deal with this situation as follows. For Regions 2, 4, and 6 we will construct smooth functions who, when concatenated with the functions on the surrounding regions, yield C^1 functions. We will then use Lemma 10.2 (stated below and proved in Appendix A of [Bel11]) to smooth these functions in an arbitrarily small neighborhood of the endpoints.

With the exceptions of R_{1414} , R_{2424} , and R_{3434} , all of the other components of the curvature tensor depend only on the functions and their first derivatives. So if we can control these three components, then we can choose δ small in Lemma 10.2 and apply the Continuity Lemma 9.4 at these endpoints. As can be seen in both equation (8.1) and Lemma 9.1, increasing any of v'' , h_θ'' , or h_r'' decreases the sectional curvature. So when bounding the curvature in each region, we will use the smallest values for the respective second derivatives in that region.

Lemma 10.2 referenced above is as follows.

Lemma 10.2. *Given real numbers k, a_1, c, a_2 with $a_1 < c < a_2$, let $f_1 : [a_1, c] \rightarrow \mathbb{R}$ and $f_2 : [c, a_2] \rightarrow \mathbb{R}$ be C^2 functions satisfying $f_i'' \geq k$, $f_1(c) = f_2(c)$, and $f_1'(c) \leq f_2'(c)$. If $f : [a_1, a_2] \rightarrow \mathbb{R}$ denotes the concatenation of f_1 and f_2 , then for any small $\delta > 0$ there exists a C^2 function $f_\delta : [a_1, a_2] \rightarrow \mathbb{R}$ such that*

- (1) $f_\delta'' > k$.
- (2) $f_\delta = f$ and $f_\delta' = f'$ at the points a_1 and a_2 .
- (3) if f is increasing, then $f_\delta' > 0$.
- (4) if f is C^l on $[a_1, a_2]$ for some integer $l \in [0, \infty]$, then f_δ is C^l on $[a_1, a_2]$, and f_δ converges to f in the C^l -topology on $[a_1, a_2]$ as $\delta \rightarrow 0$.

Lastly, one added bonus to the method that we are applying at the endpoints is that it greatly simplifies the exposition of Section 11.

10.1. Simultaneously warping h_θ and h_r from $e^{\frac{r}{2}}$ to $\cosh\left(\frac{r}{2}\right)$. Both this and the following Subsection are nearly identical to those in Section 10 of [Bel11]. The only minor changes are due to the fact that the component of the complex hyperbolic metric g with respect to $\frac{\partial}{\partial \theta}$ is $\sinh\left(\frac{r}{2}\right)$ instead of $\sinh(r)$.

Let r_ε denote the unique solution to the equation $\varepsilon e^r = \sinh\left(\frac{r}{2}\right)$. Then one sees that $r_\varepsilon \approx 2\varepsilon$. Let $r_\varepsilon^- = r_\varepsilon - \varepsilon^4$, and define $b_\varepsilon = \frac{r_\varepsilon^-}{2}$. So notice that $b_\varepsilon = \varepsilon + O(\varepsilon^3)$ (and please see Remark 10.6 for how we will use the “ $O()$ ” notation). The tangent line to the graph of $\cosh\left(\frac{r}{2}\right)$ at b_ε is

$$\ell(r) = \cosh\left(\frac{b_\varepsilon}{2}\right) + \frac{1}{2} \sinh\left(\frac{b_\varepsilon}{2}\right) (r - b_\varepsilon).$$

Let $q(r) = \ell(r) + \varepsilon^6(r - b_\varepsilon)^2$, and notice that $q(b_\varepsilon) = \ell(b_\varepsilon) = \cosh\left(\frac{b_\varepsilon}{2}\right)$.

Proposition 10.3. *There exists a C^1 function h and values $a_\varepsilon < m_\varepsilon < b_\varepsilon$ such that*

- (1) *The function h is positive and increasing.*
- (2) *$h(r) = \cosh\left(\frac{r}{2}\right)$ for $r \geq b_\varepsilon$.*
- (3) *$h(r) = q(r)$ for $r \in [m_\varepsilon, b_\varepsilon]$.*
- (4) *If $r \in [a_\varepsilon, m_\varepsilon]$, then h is smooth, $h'' > h/4$, and $(\ln h)'' > 0$ with $(\ln h)' = \frac{h'}{h} \in \left[\frac{1}{2}, \frac{3}{4}\right]$.*
- (5) *If $r \leq a_\varepsilon$, then $h(r) = e^{\frac{r}{2}}$.*

Proposition 10.3 is identical to Proposition 10.4 of [Bel11], and therefore its proof is omitted here. One quick remark is that the value of a_ε is approximately $-\frac{4}{\varepsilon}$.

10.2. Bending v from εe^r to $\sinh\left(\frac{r}{2}\right)$. First, recall the definitions of r_ε and r_ε^- from the beginning of the previous Subsection. Let $c_\varepsilon = r_\varepsilon^-$. Since the functions εe^r and $\sinh\left(\frac{r}{2}\right)$ intersect at r_ε , their concatenation yields a C^0 function. The following Proposition just says that we can approximate this by a C^1 function whose first two derivatives are controlled nicely.

Proposition 10.4. *There exists a C^1 function v and $d_\varepsilon \in (r_\varepsilon, r_\varepsilon + \varepsilon^4]$ such that*

- (1) *The function v is positive and increasing.*
- (2) *$v(r) = \sinh\left(\frac{r}{2}\right)$ for $r \geq d_\varepsilon$.*
- (3) *$v(r) = \varepsilon e^r$ for $r \leq c_\varepsilon$.*
- (4) *If $r \in [c_\varepsilon, d_\varepsilon]$, then v is smooth, $v'' > v$, and $(\ln v)'' > 0$.*

Remark 10.5. Proposition 10.4 is equivalent to Proposition 10.1 of [Bel11]. But our situation is slightly different. We are bending εe^r to the function $\sinh\left(\frac{r}{2}\right)$, not $\sinh(r)$. But one can check that the same proof works with the obvious modifications for our situation. Every derivative of either $\sinh\left(\frac{r}{2}\right)$ or $\cosh\left(\frac{r}{2}\right)$ inserts an extra “ $\frac{1}{2}$ ”, but this does not change any of the inequalities within the proof.

So in particular, we still have that $\mathbf{v}'' > \mathbf{v}$ on the interval $(c_\varepsilon, d_\varepsilon)$, even though this inequality is not true for $\sinh(\frac{r}{2})$. The function \mathbf{v} in Propostion 10.4 is only C^1 , and this shows that it is certainly not C^2 at d_ε .

10.3. Warping h_r from $\cosh(\frac{r}{2})$ to $\cosh(r)$. Let $e_\varepsilon = 2d_\varepsilon$, and let $f_\varepsilon = (k+1)e_\varepsilon$ where k is a large positive constant to be chosen independent of ε . Specifically, we first choose k large and then choose ε small, so that $f_\varepsilon = (k+1)e_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Define $\varphi : [e_\varepsilon, f_\varepsilon] \rightarrow \mathbb{R}$ by

$$(10.1) \quad \varphi(r) = C_3(r - e_\varepsilon)^3 + C_2(r - e_\varepsilon)^2 + \frac{1}{2} \sinh\left(\frac{e_\varepsilon}{2}\right)(r - e_\varepsilon) + \cosh\left(\frac{e_\varepsilon}{2}\right)$$

where

$$\begin{aligned} C_3 &= \frac{1}{\delta^2} \left(\sinh(f_\varepsilon) + \frac{1}{2} \sinh\left(\frac{e_\varepsilon}{2}\right) \right) + \frac{1}{\delta^3} \left(2 \cosh\left(\frac{e_\varepsilon}{2}\right) - 2 \cosh(f_\varepsilon) \right) \\ C_2 &= -\frac{1}{\delta} \left(\sinh(f_\varepsilon) + \sinh\left(\frac{e_\varepsilon}{2}\right) \right) + \frac{3}{\delta^2} \left(\cosh(f_\varepsilon) - \cosh\left(\frac{e_\varepsilon}{2}\right) \right) \\ \delta &= f_\varepsilon - e_\varepsilon = (k+1)e_\varepsilon - e_\varepsilon = ke_\varepsilon. \end{aligned}$$

One can check that

$$(10.2) \quad \varphi(e_\varepsilon) = \cosh\left(\frac{e_\varepsilon}{2}\right) \quad (10.4) \quad \varphi(f_\varepsilon) = \cosh(f_\varepsilon)$$

$$(10.3) \quad \varphi'(e_\varepsilon) = \frac{1}{2} \sinh\left(\frac{e_\varepsilon}{2}\right) \quad (10.5) \quad \varphi'(f_\varepsilon) = \sinh(f_\varepsilon)$$

and so one sees that φ is just the cubic polynomial which gives a C^1 interpolation between $\cosh(\frac{r}{2})$ and $\cosh(r)$ on the interval $[e_\varepsilon, f_\varepsilon]$.

The definitions for the constants C_2 and C_3 are reasonably complicated and can be difficult to use (especially when combined with the formulas in Theorem 7.1). But since $e_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we can use the Taylor series for $\sinh(r)$ and $\cosh(r)$ about 0 to give much simpler approximations for these constants. But first some notation.

Remark 10.6. We will use the “big O ” notation as follows. A term is of the order $O(\beta)$ if, when divided by β , this term still approaches 0 as $\beta \rightarrow 0$. We will generally use notation like “ $O(e_\varepsilon^2)$ ”, which just means that the remaining terms all contain degree three powers (or higher) of e_ε . Note that $O(1)$ just means that the remaining terms all contain at least one e_ε . We reserve “ \approx ” for approximations when taking k large.

With this notation, our estimates for C_2 and C_3 are as follows:

$$\begin{aligned} C_2 &= -\frac{1}{ke_\varepsilon} \left(f_\varepsilon + \frac{e_\varepsilon}{2} + O(e_\varepsilon^2) \right) + \frac{3}{k^2 e_\varepsilon^2} \left(1 + \frac{f_\varepsilon^2}{2} - 1 - \frac{e_\varepsilon^2}{8} + O(e_\varepsilon^3) \right) \\ &= -\frac{1}{ke_\varepsilon} \left(\frac{(2k+3)e_\varepsilon}{2} \right) + \frac{3}{k^2 e_\varepsilon^2} \left(\frac{(4k^2 + 8k + 3)e_\varepsilon^2}{8} \right) + O(e_\varepsilon) \\ &= \frac{-4k(2k+3) + 12k^2 + 24k + 9}{8k^2} + O(e_\varepsilon) = \frac{4k^2 + 12k + 9}{8k^2} + O(e_\varepsilon) \end{aligned}$$

$$(10.6) \quad \approx \frac{1}{2} + O(e_\varepsilon) \quad (\text{for } k \text{ large})$$

and

$$\begin{aligned} C_3 &\approx \frac{1}{k^2 e_\varepsilon^2} \left(f_\varepsilon + \frac{e_\varepsilon}{4} + O(e_\varepsilon^2) \right) + \frac{1}{k^3 e_\varepsilon^3} \left(2 + \frac{e_\varepsilon^2}{4} - 2 - f_\varepsilon^2 + O(e_\varepsilon^3) \right) \\ &= \frac{1}{k^2 e_\varepsilon^2} \left(\frac{(4k+5)e_\varepsilon}{4} \right) + \frac{1}{k^3 e_\varepsilon^3} \left(\frac{(-4k^2 - 8k - 3)e_\varepsilon^2}{4} \right) + O(1) \\ &= \frac{-3k-3}{4k^3 e_\varepsilon} + O(1) = \frac{-3(k+1)}{4k^3 e_\varepsilon} + O(1) \\ (10.7) \quad &\approx \frac{-3}{4e_\varepsilon k^2} + O(1) \quad (\text{for } k \text{ large}). \end{aligned}$$

Now, for $r \in [e_\varepsilon, f_\varepsilon]$, we can write $r = Ce_\varepsilon$ for some $1 \leq C \leq k+1$. Then letting $\alpha = C - 1$ we have that $r - e_\varepsilon = e_\varepsilon(C - 1) = \alpha e_\varepsilon$ where, of course, $0 \leq \alpha \leq k$. Using this notation with the approximations in equations (10.6) and (10.7) turns equation (10.1) into

$$\begin{aligned} \varphi &\approx \left(\frac{-3}{4e_\varepsilon k^2} + O(1) \right) (\alpha^3 e_\varepsilon^3) + \left(\frac{1}{2} + O(e_\varepsilon) \right) (\alpha^2 e_\varepsilon^2) \\ &\quad + \frac{1}{2} \left(\frac{e_\varepsilon}{2} + O(e_\varepsilon^2) \right) (\alpha e_\varepsilon) + \left(1 + \frac{e_\varepsilon^2}{8} + O(e_\varepsilon^3) \right) \\ (10.8) \quad &= 1 + e_\varepsilon^2 \left(\frac{-3}{4k^2} \alpha^3 + \frac{1}{2} \alpha^2 + \frac{1}{4} \alpha + \frac{1}{8} \right) + O(e_\varepsilon^3). \end{aligned}$$

In Section 11 we will also need approximations for φ' and φ'' . So let us compute them now.

$$\begin{aligned} \varphi' &= 3C_3(r - e_\varepsilon)^2 + 2C_2(r - e_\varepsilon) + \frac{1}{2} \sinh \left(\frac{e_\varepsilon}{2} \right) \\ &\approx \left(\frac{-9}{4e_\varepsilon k^2} + O(1) \right) (\alpha^2 e_\varepsilon^2) + (1 + O(e_\varepsilon))(\alpha e_\varepsilon) + \left(\frac{e_\varepsilon}{4} + O(e_\varepsilon^2) \right) \\ (10.9) \quad &= e_\varepsilon \left(\frac{-9}{4k^2} \alpha^2 + \alpha + \frac{1}{4} \right) + O(e_\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned} \varphi'' &= 6C_3(r - e_\varepsilon) + 2C_2 \\ &\approx \left(\frac{-9}{2e_\varepsilon k^2} + O(1) \right) (\alpha e_\varepsilon) + (1 + O(e_\varepsilon)) \\ (10.10) \quad &= \frac{-9}{2k^2} \alpha + 1 + O(e_\varepsilon) \approx 1 + O(e_\varepsilon) \end{aligned}$$

since $0 \leq \alpha \leq k$ and k is large.

11. PROVING THE METRIC IN SECTION 10 HAS SECTIONAL CURVATURE BOUNDED ABOVE BY A NEGATIVE CONSTANT

In this Section we do exactly as the title says, proving Theorem 1.1(1). We break our argument up into the different regions. Note that region 7 is clear since the metric λ agrees with the complex hyperbolic metric g . Throughout this Section we again use the notation $R_{ijkl} = \langle R(Y_i, Y_j)Y_k, Y_l \rangle_\lambda$.

Region 1. Here, $v = \varepsilon e^r$ and $h_\theta = h_r = e^{\frac{r}{2}} := h$ over the region $(-\infty, a_\varepsilon)$. Plugging these values into the equations in Corollary 8.1 yields

$$\begin{aligned} R_{1212} &= R_{1313} = -\frac{1}{2} + \frac{\varepsilon}{16} \\ R_{2323} &= -\frac{1}{4} - \frac{1}{4e^r} - \frac{3\varepsilon^2}{16} \\ R_{1414} &= -1 & R_{2424} &= R_{3434} = -\frac{1}{4} \\ R_{1423} &= \frac{-\varepsilon}{4} & R_{1234} &= \frac{\varepsilon}{8} = -R_{1324}. \end{aligned}$$

Then Lemma 9.1 proves that the sectional curvature of λ is bounded above by a negative constant on this region.

Region 2. In this region $v = \varepsilon e^r$ and $h_\theta = h_r := h$ is warped from $e^{\frac{r}{2}}$ to $\cosh\left(\frac{r}{2}\right)$ over the interval $(a_\varepsilon, b_\varepsilon)$. Region 2 is broken up into two different situations: $r \in (a_\varepsilon, m_\varepsilon)$ and $r \in (m_\varepsilon, b_\varepsilon)$. We deal with the two intervals separately. Our arguments for these regions are virtually identical to those in [Bel11]. We include the arguments here because our curvature formulas in Corollary 8.1 are slightly different than Belegradek's, and to verify a few estimates in our case (since our r_ε is twice that of what is in [Bel11]).

The interval $(a_\varepsilon, m_\varepsilon)$. By Proposition 10.3 we know that $\frac{h''}{h} > \frac{1}{4}$, $\frac{h'}{h}$ is increasing (since $(\ln h)'' > 0$), and $\frac{h'}{h} \in [\frac{1}{2}, \frac{3}{4}]$. Of course $\frac{v''}{v} = \frac{v'}{v} = 1$, and so $\frac{v'}{v} - \frac{h'}{h} \leq \frac{1}{2} < 1$. Since $\ln h$ is convex, its graph lies above its tangent line at a_ε . Hence $\ln h \geq \frac{r}{2}$ implying that $h \geq e^{\frac{r}{2}}$. Therefore, $\frac{v}{h^2} \leq \frac{\varepsilon e^r}{e^r} = \varepsilon < 2\varepsilon$.

Plugging these estimates into the formulas in Corollary 8.1 yields

$$\begin{aligned} K(Y_1, Y_2) &= K(Y_1, Y_3) \leq -\frac{1}{2} + \frac{\varepsilon^2}{4} < -\frac{1}{3} \\ K(Y_2, Y_3) &< -\left(\frac{h'}{h}\right)^2 \leq -\frac{1}{4} \\ K(Y_1, Y_4) &= -1 & K(Y_2, Y_4) &= K(Y_3, Y_4) = -\frac{h''}{h} < -\frac{1}{4} \\ |R_{1423}| &= \left| \frac{-v}{2h^2} \left(\frac{v'}{v} - \frac{h'}{h} \right) \right| < \varepsilon & |R_{1234}| &= |R_{1324}| < \frac{\varepsilon}{2}. \end{aligned}$$

Lemma 9.1 then completes the argument for $(a_\varepsilon, m_\varepsilon)$.

The interval $(m_\varepsilon, b_\varepsilon)$. Over this interval $v = \varepsilon e^r$ and $h_\theta = h_r = q(r) := h$, where $q(r)$ is defined in Subsection 10.1. Two things that are immediately clear are that $\frac{v''}{v} = 1$ and $\frac{h''}{h} > \varepsilon^6$. We also have that

$$q'(r) = \frac{1}{2} \sinh\left(\frac{b_\varepsilon}{2}\right) + 2\varepsilon^6(r - b_\varepsilon) = \frac{\varepsilon}{4} + O(\varepsilon^2)$$

and so, in particular, q is increasing. Thus $q(r) < q(b_\varepsilon) = \cosh\left(\frac{b_\varepsilon}{2}\right) = 1 + O(\varepsilon^2)$. Since $q'' = 2\varepsilon^6$, one easily checks that $\frac{q'}{q}$ is decreasing over the interval $(m_\varepsilon, b_\varepsilon)$ from $\frac{3}{4}$ to $\frac{1}{2} \tanh\left(\frac{b_\varepsilon}{2}\right) = \frac{\varepsilon}{4} + O(\varepsilon^2)$. Therefore, $\frac{h'}{h} \in \left(\frac{\varepsilon}{5}, \frac{4}{5}\right)$.

The last quantity that we need to bound is $\frac{v}{h^2} = \frac{\varepsilon e^r}{q^2}$. The argument is identical to pg. 567 of [Bel11], and so we omit it here. The idea of the argument is that you differentiate $\frac{v}{h^2}$ twice to show that it is locally maximized at the endpoints $\{m_\varepsilon, b_\varepsilon\}$, and then argue that the maximum value is actually at $r = b_\varepsilon$. You then have that

$$\frac{v}{h^2} \leq \varepsilon \frac{e^{b_\varepsilon}}{\cosh^2\left(\frac{b_\varepsilon}{2}\right)} < 2\varepsilon.$$

Plugging these estimates into the equations in Corollary 8.1 gives

$$\begin{aligned} K(Y_1, Y_2) &= K(Y_1, Y_3) < -\frac{\varepsilon}{5} + \frac{\varepsilon^2}{4} < -\frac{\varepsilon}{6} \\ K(Y_2, Y_3) &< -\frac{1}{4h^2} \leq \frac{-1}{4\cosh^2\left(\frac{b_\varepsilon}{2}\right)} < -\frac{1}{9} \\ K(Y_1, Y_4) &= -1 & K(Y_2, Y_4) &= K(Y_3, Y_4) < -\varepsilon^6 \\ |R_{1423}| &= \left| \frac{-v}{2h^2} \left(\frac{v'}{v} - \frac{h'}{h} \right) \right| < \frac{1}{2} \cdot (2\varepsilon) \left(1 - \frac{\varepsilon}{5} \right) < \varepsilon. \end{aligned}$$

Then inserting these values into equation (8.1) yields

$$\begin{aligned} K(\sigma) &< -\frac{\varepsilon}{6} \left((c_1 d_2 - c_2 d_1)^2 + d_1^2 c_3^2 \right) - d_1^2 c_4^2 - \frac{1}{9} d_2^2 c_3^2 - \varepsilon^6 d_2^2 c_4^2 + 3\varepsilon |c_3 c_4 d_1 d_2| \\ &= -\frac{\varepsilon}{6} \left((c_1 d_2 - c_2 d_1)^2 + d_1^2 c_3^2 \right) - \varepsilon^6 d_2^2 c_4^2 - \left(|c_4 d_1| - \frac{1}{3} |c_3 d_2| \right)^2 \\ &\quad - \left(\frac{2}{3} - 3\varepsilon \right) |c_3 c_4 d_1 d_2|. \end{aligned}$$

Every term in the above sum is nonpositive, and not all of the coefficients can simultaneously be zero. So by compactness we have that $K()$ is bounded above by a negative constant within Region $(m_\varepsilon, b_\varepsilon)$.

Region 3. Here, $v = \varepsilon e^r$ and $h_\theta = h_r = \cosh\left(\frac{r}{2}\right) := h$ over the region $(b_\varepsilon, c_\varepsilon)$. Recall that $b_\varepsilon \approx \varepsilon$ and $c_\varepsilon \approx 2\varepsilon$. Then for $\varepsilon > 0$ small enough we have:

$$\begin{aligned} \bullet \quad \frac{v}{h^2} &= \frac{\varepsilon e^r}{\cosh^2\left(\frac{r}{2}\right)} \leq \frac{\varepsilon e^{2\varepsilon}}{1} < 2\varepsilon. \\ \bullet \quad \frac{h'}{h} &= \frac{\sinh\left(\frac{r}{2}\right)}{2\cosh\left(\frac{r}{2}\right)} \geq \frac{\frac{r}{2}}{4} = \frac{r}{8} \geq \frac{\varepsilon}{8} > \frac{\varepsilon}{9}. \end{aligned}$$

Then we can plug into the equations in Corollary 8.1 to obtain:

$$\begin{aligned} K(Y_1, Y_2) &= K(Y_1, Y_3) = -\frac{h'}{h} + \frac{v^2}{16h^4} < \frac{\varepsilon}{9} + \frac{\varepsilon^2}{4} < -\frac{\varepsilon}{10} \\ K(Y_2, Y_3) &< \frac{-1}{4h^2} < -\frac{1}{9} \\ K(Y_1, Y_4) &= -1 & K(Y_2, Y_4) &= K(Y_3, Y_4) = -\frac{1}{4} \\ |R_{1423}| &= \frac{1}{2} \frac{v}{h^2} \left(\ln \frac{v}{h} \right)' \leq \frac{1}{2} (2\varepsilon) \left(\frac{v'}{v} - \frac{h'}{h} \right) = \frac{3}{4} \varepsilon < \varepsilon. \end{aligned}$$

To find $K(\sigma)$ for any 2-plane σ we plug these values into equation (8.1), giving

$$\begin{aligned} K(\sigma) &\leq -\frac{\varepsilon}{10} ((c_1 d_2 - c_2 d_1)^2 + d_1^2 c_3^2) - \frac{1}{4} (d_1^2 c_4^2 + d_2^2 c_4^2) - \frac{1}{9} d_2^2 c_3^2 + 3\varepsilon |c_3 c_4 d_1 d_2| \\ &= -\frac{\varepsilon}{10} ((c_1 d_2 - c_2 d_1)^2 + d_1^2 c_3^2) - \frac{1}{4} d_2^2 c_4^2 - \left(\frac{1}{2} |c_4 d_1| - \frac{1}{3} |c_3 d_2| \right)^2 \\ &\quad + |c_3 c_4 d_1 d_2| \left(3\varepsilon - \frac{1}{3} \right). \end{aligned}$$

Every term in the sum above is nonpositive. So we have that $K()$ is bounded above by a negative constant within Region 3.

Region 4. In this region $h_\theta = h_r = \cosh\left(\frac{r}{2}\right)$ over the region $(c_\varepsilon, d_\varepsilon)$, while v is “smoothed out” from εe^r to $\sinh\left(\frac{r}{2}\right)$. Recall from Proposition 10.4 that over this region v and v' are increasing, $v'' \geq \frac{v}{4}$, and $(\ln v)'' > 0$ (which, in particular, implies that $\frac{v'}{v}$ is increasing). Also, recall that $c_\varepsilon = r_\varepsilon^- = r_\varepsilon - \varepsilon^4 \approx 2\varepsilon - \varepsilon^4$ and $d_\varepsilon \leq r_\varepsilon + \varepsilon^4 \approx 2\varepsilon + \varepsilon^4$. So over this entire region, $r \approx 2\varepsilon$.

Since both v and h are increasing we have that

$$\frac{v}{h^2} \leq \frac{v(d_\varepsilon)}{h(c_\varepsilon)} \approx \frac{\sinh\left(\frac{2\varepsilon + \varepsilon^4}{2}\right)}{\cosh\left(\frac{2\varepsilon - \varepsilon^4}{2}\right)} \approx \frac{\frac{2\varepsilon}{2}}{1} = \varepsilon < 2\varepsilon.$$

We also know that $\frac{v'}{v}$ is increasing, and so it can be bounded by its values at the endpoints of the interval (where $v = \varepsilon e^r$ and $\sinh\left(\frac{r}{2}\right)$, respectively). Therefore,

$$1 \leq \frac{v'}{v} \leq \frac{1}{2} \coth\left(\frac{d_\varepsilon}{2}\right)$$

which, for $\varepsilon > 0$ small, is very large. We also have that

$$\frac{h'}{h} = \frac{\sinh\left(\frac{r}{2}\right)}{2 \cosh\left(\frac{r}{2}\right)} = \frac{1}{2} \tanh\left(\frac{r}{2}\right) \approx \frac{r}{4} \approx \frac{2\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Combining these last two equations gives us

- $0 < \frac{v'}{v} - \frac{h'}{h} < \frac{1}{2} \coth\left(\frac{d_\varepsilon}{2}\right)$
- $\frac{h'v'}{hv} > \frac{\varepsilon}{4}$
- $\frac{v}{h^2} \left(\frac{v'}{v} - \frac{h'}{h} \right) \leq \frac{\sinh\left(\frac{d_\varepsilon}{2}\right)}{\cosh\left(\frac{c_\varepsilon}{2}\right)} \cdot \frac{1}{2} \coth\left(\frac{d_\varepsilon}{2}\right) = \frac{\cosh\left(\frac{d_\varepsilon}{2}\right)}{2 \cosh\left(\frac{c_\varepsilon}{2}\right)} = \frac{1}{2} + O(\varepsilon^2).$

Combining these estimates with the formulas in Corollary 8.1 gives us

$$\begin{aligned}
 K(Y_1, Y_2) &= K(Y_1, Y_3) = -\frac{v'h'}{vh} + \frac{v^2}{16h^4} < -\frac{\varepsilon}{4} + \frac{\varepsilon^2}{4} < -\frac{\varepsilon}{5} \\
 K(Y_2, Y_3) &< -\frac{1}{4h^2} - \left(\frac{h'}{h}\right)^2 = \frac{-1 - \sinh^2\left(\frac{r}{2}\right)}{4 \cosh^2\left(\frac{r}{2}\right)} = -\frac{1}{4} \\
 K(Y_1, Y_4) &< -1 & K(Y_2, Y_4) &= K(Y_3, Y_4) = -\frac{1}{4} \\
 |R_{1423}| &= \left| \frac{v}{2h^2} \left(\frac{v'}{v} - \frac{h'}{h} \right) \right| < \frac{1}{4} + \varepsilon.
 \end{aligned}$$

To find $K(\sigma)$ for any 2-plane σ we plug these values into equation (8.1), giving

$$\begin{aligned}
 K(\sigma) &< -\frac{\varepsilon}{5} \left((c_1d_2 - c_2d_1)^2 + c_3^2d_1^2 \right) - c_4^2d_1^2 - \frac{1}{4}(c_3^2d_2^2 + c_4^2d_2^2) + 3 \left(\frac{1}{4} + \varepsilon \right) |c_3c_4d_1d_2| \\
 &= -\frac{\varepsilon}{5} \left((c_1d_2 - c_2d_1)^2 + d_1^2c_3^2 \right) - \left(c_4d_1 - \frac{1}{2}d_2c_3 \right)^2 - \frac{1}{4}c_4^2d_2^2 \\
 &\quad - \left(\frac{5}{4} - \varepsilon \right) |c_3c_4d_1d_2|.
 \end{aligned}$$

Every term in the sum above is nonpositive, and so we have that $K()$ is bounded above by a negative constant within Region 4.

Region 5. Here, $v = \sinh\left(\frac{r}{2}\right)$ and $h_\theta = h_r = \cosh\left(\frac{r}{2}\right) := h$ over the region $(d_\varepsilon, e_\varepsilon)$. Plugging these values into the equations in Corollary 8.1 yields

$$\begin{aligned}
 R_{1212} &= R_{1313} = -\frac{1}{4} + \frac{\sinh^2\left(\frac{r}{2}\right)}{16 \cosh^4\left(\frac{r}{2}\right)} \approx -\frac{1}{4} \\
 R_{2323} &= -\frac{1}{4} - \frac{3 \sinh^2\left(\frac{r}{2}\right)}{16 \cosh^4\left(\frac{r}{2}\right)} < -\frac{1}{4} \\
 R_{1414} &= R_{2424} = R_{3434} = -\frac{1}{4} \\
 R_{1423} &= \frac{-1}{4 \cosh^3\left(\frac{r}{2}\right)} > -\frac{1}{4} & R_{1234} &= -R_{1324} = \frac{1}{8 \cosh^3\left(\frac{r}{2}\right)} \approx \frac{1}{8}.
 \end{aligned}$$

Then for $\varepsilon > 0$ small enough, Lemma 9.1 proves that the sectional curvature of λ is bounded above by a negative constant on this region.

Region 6. In this region $v = \sinh\left(\frac{r}{2}\right)$, $h_\theta = \cosh\left(\frac{r}{2}\right)$, and $h_r = \varphi$ where φ is the cubic defined in Subsection 10.3 which varies in a C^1 manner between $\cosh\left(\frac{r}{2}\right)$ and $\cosh(r)$. If one plugs in the values $v = \sinh\left(\frac{r}{2}\right)$, $h_\theta = \cosh\left(\frac{r}{2}\right)$ and $h_r = \varphi$ into the equations in Theorem 7.1 and then simplifies (a lot), they come up with the

following.

$$\begin{aligned}
R_{1212} &= \frac{-(\varphi^2 - 1)(3\varphi^2 + \cosh^2(r))}{4\varphi^2 \sinh^2(r)} \\
R_{1313} &= \frac{-(\cosh(r) + 1)\varphi'}{2 \sinh(r)\varphi} - \frac{1}{4\varphi^2} + \frac{(\varphi^2 - 1)(\varphi^2 + 2 \cosh(r) + 1)}{4\varphi^2 \sinh^2(r)} \\
R_{2323} &= \frac{-(\cosh(r) - 1)\varphi'}{2 \sinh(r)\varphi} - \frac{1}{4\varphi^2} + \frac{(\varphi^2 - 1)(\varphi^2 - 2 \cosh(r) + 1)}{4\varphi^2 \sinh^2(r)} \\
R_{1414} &= -\frac{1}{4} \quad R_{2424} = -\frac{1}{4} \quad R_{3434} = -\frac{\varphi''}{\varphi} \\
R_{1234} &= \frac{\varphi'}{\sinh(r)} - \frac{(\varphi^2 - 1) \cosh(r)}{2\varphi \sinh^2(r)} \\
R_{1324} &= \frac{\varphi'(\cosh(r) + \varphi^2)}{2\varphi^2 \sinh(r)} - \frac{(\varphi^2 - 1)(\cosh(r) + 1)}{4\varphi \sinh^2(r)} - \frac{1}{4\varphi} \\
R_{1423} &= \frac{\varphi'(\cosh(r) - \varphi^2)}{2\varphi^2 \sinh(r)} + \frac{(\varphi^2 - 1)(\cosh(r) - 1)}{4\varphi \sinh^2(r)} - \frac{1}{4\varphi}.
\end{aligned}$$

One nice way to “check” these formulas is to let $\varphi = \cosh(r)$ and confirm that this gives you the values in equations (4.1) through (4.9).

Our method to prove that $K()$ is bounded above by a negative constant in region 6 is to attempt to apply Lemma 9.1, and when exactly one of these inequalities fails by an arbitrarily small amount to really apply Corollary 9.5. So we need to consider the inequalities in Lemma 9.1. But first we compute two estimates (equations (11.1) and (11.2)) which show up in many of the inequalities.

The Taylor series for $\sinh(r)$ centered at $r = e_\varepsilon$ is

$$\begin{aligned}
\sinh(r) &= \sinh(e_\varepsilon) + \cosh(e_\varepsilon)(r - e_\varepsilon) + \frac{\sinh(e_\varepsilon)}{2}(r - e_\varepsilon)^2 + \dots \\
&= \left(e_\varepsilon + \frac{e_\varepsilon^3}{6}\right) + \left(1 + \frac{e_\varepsilon^2}{2}\right)e_\varepsilon\alpha + \frac{1}{2}\left(e_\varepsilon + \frac{e_\varepsilon^3}{6}\right)e_\varepsilon^2\alpha^2 + O(e_\varepsilon^2) \\
&= e_\varepsilon(1 + \alpha) + O(e_\varepsilon^2).
\end{aligned}$$

Combining this with equation (10.9) gives

$$(11.1) \quad \frac{\varphi'(r)}{\sinh(r)} \approx \frac{e_\varepsilon \left(\frac{1}{4} + \alpha - \frac{9}{4k^2}\alpha^2\right) + O(e_\varepsilon^2)}{e_\varepsilon(1 + \alpha) + O(e_\varepsilon^2)} = \frac{\frac{1}{4} + \alpha - \frac{9}{4k^2}\alpha^2}{1 + \alpha} + O(e_\varepsilon).$$

Now, using equation (10.8) one sees that

$$\varphi^2(r) - 1 \approx e_\varepsilon^2 \left(\frac{1}{4} + \frac{1}{2}\alpha + \alpha^2 - \frac{3}{2k^2}\alpha^3\right) + O(e_\varepsilon^3).$$

Also, the above formula for $\sinh(r)$ gives us that

$$\sinh^2(r) = e_\varepsilon^2(1 + \alpha)^2 + O(e_\varepsilon^3).$$

Therefore

$$(11.2) \quad \frac{\varphi^2(r) - 1}{\sinh^2(r)} = \frac{\frac{1}{4} + \frac{1}{2}\alpha + \alpha^2 - \frac{3}{2k^2}\alpha^3}{(1 + \alpha)^2} + O(e_\varepsilon).$$

We now deal with each of the six inequalities in Lemma 9.1. We consider them in reverse order since the inequality that will cause issues is inequality (1a), and we will prove that each inequality (except (1a)) is strict. Note that we will have to derive estimates other than (11.1) and (11.2) in some of these situations, but these two come up so often that we did them first.

Inequality (3b). Inequality (3b) is $R_{2323} < -|R_{1423}|$. Over region 6 we have that $R_{1423} < 0$, and so we show that $R_{2323} < R_{1423}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} \varphi - 1 &< 2 \left(\frac{\varphi'}{\sinh(r)} \right) (1 + \varphi)(\cosh(r) - \varphi) \\ &\quad - \left(\frac{\varphi^2 - 1}{\sinh^2(r)} \right) (\varphi^2 - 2 \cosh(r) + 1 - \varphi \cosh(r) + \varphi) \end{aligned}$$

To prove the above inequality, we need the following estimates:

$$\begin{aligned} \cosh(r) &= 1 + e_\varepsilon^2 \left(\frac{1}{2} + \alpha + \frac{1}{2} \alpha^2 \right) + O(e_\varepsilon^3) \\ \varphi^2(r) &\approx 1 + e_\varepsilon^2 \left(\frac{1}{4} + \frac{1}{2} \alpha + \alpha^2 - \frac{3}{2k^2} \alpha^3 \right) + O(e_\varepsilon^3) \\ \cosh(r) - \varphi(r) &\approx e_\varepsilon^2 \left(\frac{3}{8} + \frac{3}{4} \alpha + \frac{3}{4k^2} \alpha^3 + O(e_\varepsilon) \right) \\ \varphi^2(r) - 2 \cosh(r) + 1 - \varphi \cosh(r) + \varphi &\approx e_\varepsilon^2 \left(-\frac{5}{4} - \frac{5}{2} \alpha - \frac{1}{2} \alpha^2 - \frac{3}{2k^2} \alpha^3 + O(e_\varepsilon) \right). \end{aligned}$$

Then, applying equations (11.1) and (11.2) with the above estimates, we have that inequality (3b) holds if and only if

$$\begin{aligned} e_\varepsilon^2 \left(\frac{1}{8} + \frac{1}{4} \alpha + \frac{1}{2} \alpha^2 - \frac{3}{4k^2} \alpha^3 \right) &< e_\varepsilon^2 \left(\frac{\frac{1}{4} + \alpha - \frac{9}{4k^2} \alpha^2}{1 + \alpha} \right) \left(\frac{3}{2} + 3\alpha + \frac{3}{k^2} \alpha^3 \right) \\ &\quad + e_\varepsilon^2 \left(\frac{\frac{1}{4} + \frac{1}{2} \alpha + \alpha^2 - \frac{3}{2k^2} \alpha^3}{(1 + \alpha)^2} \right) \left(\frac{5}{4} + \frac{5}{2} \alpha + \frac{1}{2} \alpha^2 + \frac{3}{2k^2} \alpha^3 \right) + O(e_\varepsilon^3). \end{aligned}$$

This inequality is true if and only if

$$\begin{aligned} O(e_\varepsilon) &< \frac{9}{16} + \frac{27}{8} \alpha + \left(\frac{27}{4} - \frac{9}{4k^2} \right) \alpha^2 + \left(\frac{9}{2} - \frac{9}{k^2} \right) \alpha^3 - \frac{3}{4k^2} \alpha^4 \\ &\quad + \left(\frac{18}{4k^2} - \frac{27}{4k^4} \right) \alpha^5 - \frac{9}{k^4} \alpha^6 \\ \iff O(e_\varepsilon) &< \frac{9}{16} + \frac{27}{8} \alpha + \left(\frac{27}{4} - \frac{9}{4k^2} \right) \alpha^2 + \left(\frac{9}{4} - \frac{9}{k^2} \right) \alpha^3 + \left(\frac{9}{4} - \frac{3\alpha}{4k^2} \right) \alpha^3 \\ &\quad + \left(\frac{9}{4k^2} - \frac{27}{4k^4} \right) \alpha^5 + \left(\frac{9}{4k^2} - \frac{9\alpha}{k^4} \right) \alpha^5. \end{aligned}$$

For k large enough, every term on the right hand side of the above equation is positive (since $0 \leq \alpha \leq k$). So the right hand side is bounded below by $\frac{9}{16}$. This proves that inequality (3b) holds for ε sufficiently small.

Inequality (3a). Inequality (3a) is $R_{1414} < -|R_{1423}|$. Over region 6 we have that $R_{1423} < 0$, and so we show that $R_{1414} < R_{1423}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} \frac{\cosh(r) - \varphi^2}{\varphi(1 - \varphi)} &< \frac{\sinh(r)}{\varphi'} \left(\frac{\varphi + \cosh(r) + 2}{2(\cosh(r) + 1)} \right) \\ &= \frac{\sinh(r)}{\varphi'} + O(e_\varepsilon). \end{aligned}$$

To prove the above inequality we need the following estimates:

$$\cosh(r) - \varphi^2(r) \approx e_\varepsilon^2 \left(\frac{1}{4} + \frac{1}{2}\alpha - \frac{1}{2}\alpha^2 + \frac{3}{2k^2}\alpha^3 + O(e_\varepsilon) \right)$$

and

$$\varphi(r)(1 - \varphi(r)) \approx e_\varepsilon^2 \left(-\frac{1}{8} - \frac{1}{4}\alpha - \frac{1}{2}\alpha^2 + \frac{3}{4k^2}\alpha^3 + O(e_\varepsilon) \right).$$

These can be obtained by using the estimates derived for inequality (3b) along with equation (10.8). Then, using the above estimates with equation (11.1), inequality (3a) is satisfied if and only if

$$\begin{aligned} \frac{\frac{1}{4} + \frac{1}{2}\alpha - \frac{1}{2}\alpha^2 + \frac{3}{2k^2}\alpha^3}{-\frac{1}{8} - \frac{1}{4}\alpha - \frac{1}{2}\alpha^2 + \frac{3}{4k^2}\alpha^3} &< \frac{1 + \alpha}{\frac{1}{4} + \alpha - \frac{9}{4k^2}\alpha^2} + O(e_\varepsilon) \\ \iff O(e_\varepsilon) &< 6 + 24\alpha + \left(36 - \frac{18}{k^2} \right) \alpha^2 - \frac{48}{k^2} \alpha^3 + \frac{60}{k^2} \alpha^4 - \frac{108}{k^4} \alpha^5 \\ \iff O(e_\varepsilon) &< 6 + 24\alpha + \left(18 - \frac{18}{k^2} \right) \alpha^2 + \left(18 - \frac{48}{k^2} \alpha \right) \alpha^2 + \frac{1}{k^2} \left(60 - \frac{108}{k^2} \alpha \right) \alpha^4. \end{aligned}$$

Recall that $0 \leq \alpha \leq k$. So every term on the right hand side of the last inequality is positive for k sufficiently large. Therefore the right hand side of the inequality is bounded below by 6, which verifies that inequality (3a) is satisfied for ε sufficiently small.

Inequality (2b). Inequality (2b) is $R_{2424} < -|R_{1324}|$. Over region 6 we have that R_{1324} varies from $\approx -\frac{1}{8}$ to $\frac{1}{4}$. We will show that $R_{2424} < -R_{1324}$, and an analogous (simpler) argument shows that $R_{1414} < R_{1324}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} -\frac{1}{4} &< -\frac{\varphi'(\cosh(r) + \varphi^2)}{2\varphi^2 \sinh(r)} + \frac{(\varphi^2 - 1)(\cosh(r) + 1)}{4\varphi \sinh^2(r)} + \frac{1}{4\varphi} \\ &= -\frac{\varphi'}{\sinh(r)} + \frac{(\varphi^2 - 1)}{2\sinh^2(r)} + \frac{1}{4} + O(e_\varepsilon). \end{aligned}$$

Using equations (11.1) and (11.2), we see that inequality (2b) is satisfied if and only if

$$\begin{aligned} -\frac{1}{2} + O(e_\varepsilon) &< -\left(\frac{\frac{1}{4} + \alpha - \frac{9}{4k^2}\alpha^2}{1 + \alpha} \right) + \frac{1}{2} \left(\frac{\frac{1}{4} + \frac{1}{2}\alpha + \alpha^2 - \frac{3}{2k^2}\alpha^3}{(1 + \alpha)^2} \right) \\ \iff -(1 + \alpha)^2 + O(e_\varepsilon) &< -\frac{1}{4} - 2\alpha + \left(\frac{9}{2k^2} - 1 \right) \alpha^2 + \frac{3}{k^2} \alpha^3 \\ \iff O(e_\varepsilon) &< \frac{3}{4} + \frac{9}{2k^2} \alpha^2 + \frac{3}{k^2} \alpha^3. \end{aligned}$$

The right hand side of this inequality is clearly bounded below by $\frac{3}{4}$ since $0 \leq \alpha \leq k$. Thus, for ε sufficiently small, inequality (2b) is satisfied.

Inequality (2a). Inequality (2a) is $R_{1313} < -|R_{1324}|$. Over region 6 we have that R_{1324} varies from $\approx -\frac{1}{8}$ to $\frac{1}{4}$. We will show that $R_{1313} < -R_{1324}$, and an analogous (simpler) argument shows that $R_{1313} < R_{1324}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} -\frac{(\varphi+1)}{4\varphi^2} &< \frac{\varphi'(\varphi + \varphi \cosh(r) - \cosh(r) - \varphi^2)}{2\varphi^2 \sinh(r)} \\ &\quad + \frac{(\varphi^2 - 1)(\varphi \cosh(r) + \varphi - \varphi^2 - 2 \cosh(r) - 1)}{4\varphi^2 \sinh^2(r)} \\ \iff -\frac{1}{2} + O(e_\varepsilon) &< \left(\frac{\varphi'}{2 \sinh(r)} \right) (\varphi + \varphi \cosh(r) - \cosh(r) - \varphi^2) - \frac{(\varphi^2 - 1)}{2 \sinh^2(r)}. \end{aligned}$$

But one can check that

$$\varphi + \varphi \cosh(r) - \cosh(r) - \varphi^2 = O(e_\varepsilon)$$

and therefore what we need to show is that

$$\frac{(\varphi^2 - 1)}{\sinh^2(r)} + O(e_\varepsilon) < 1.$$

Using equation (11.2), we have that inequality (2a) is satisfied if and only if

$$\begin{aligned} \frac{\frac{1}{4} + \frac{1}{2}\alpha + \alpha^2 - \frac{3}{2k^2}\alpha^3}{(1 + \alpha)^2} + O(e_\varepsilon) &< 1 \\ \iff O(e_\varepsilon) &< \frac{3}{4} + \frac{3}{2}\alpha + \frac{3}{2k^2}\alpha^3. \end{aligned}$$

The right hand side is clearly bounded below by $\frac{3}{4}$ for $0 \leq \alpha \leq k$, proving that inequality (2a) is satisfied for ε sufficiently small.

Inequality (1b). Inequality (1b) is $R_{3434} < -|R_{1234}|$. Over region 6 we have that $R_{1234} > 0$, and so we will show that $R_{3434} < -R_{1234}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} -\frac{\varphi''}{\varphi} &< -\frac{\varphi'}{\sinh(r)} + \frac{(\varphi^2 - 1) \cosh(r)}{2\varphi \sinh^2(r)} \\ \iff -1 + O(e_\varepsilon) &< -\frac{\varphi'}{\sinh(r)} + \frac{\varphi^2 - 1}{2 \sinh^2(r)} + O(e_\varepsilon) \\ \iff -1 + O(e_\varepsilon) &< -\frac{\varphi'}{\sinh(r)} + \frac{\varphi^2 - 1}{2 \sinh^2(r)}. \end{aligned}$$

Note that this inequality is weaker than inequality (2b). For (2b), we had a “ $-\frac{1}{2}$ ” instead of a “ -1 ” on the left hand side of the inequality. Therefore, since inequality (2b) holds for ε sufficiently small, so does inequality (1b).

Inequality (1a). Inequality (1a) is $R_{1212} < -|R_{1234}|$. Over region 6 we have that $R_{1234} > 0$, and so we will (try to) show that $R_{1212} < -R_{1234}$. Using the equations above, we see that this inequality holds if and only if

$$\begin{aligned} & \frac{-(\varphi^2 - 1)(3\varphi^2 + \cosh^2(r))}{4\varphi^2 \sinh^2(r)} < -\frac{\varphi'}{\sinh(r)} + \frac{(\varphi^2 - 1) \cosh(r)}{2\varphi \sinh^2(r)} \\ \iff & -\frac{(\varphi^2 - 1)}{\sinh^2(r)} + O(e_\varepsilon) < -\frac{\varphi'}{\sinh(r)} + \frac{\varphi^2 - 1}{2 \sinh^2(r)} + O(e_\varepsilon) \\ \iff & \frac{\varphi'}{\sinh(r)} + O(e_\varepsilon) < \frac{3}{2} \left(\frac{\varphi^2 - 1}{\sinh^2(r)} \right). \end{aligned}$$

Using the above estimates, inequality (1a) is satisfied if and only if

$$\begin{aligned} & \frac{\frac{1}{4} + \alpha - \frac{9}{4k^2}\alpha^2}{1 + \alpha} + O(e_\varepsilon) < \frac{3}{2} \left(\frac{\frac{1}{4} + \frac{1}{2}\alpha + \alpha^2 - \frac{3}{2k^2}\alpha^3}{(1 + \alpha)^2} \right) \\ \iff & O(e_\varepsilon) < \frac{1}{4} - \alpha + \left(1 + \frac{9}{2k^2} \right) \alpha^2. \end{aligned}$$

Let $p(\alpha) = \frac{1}{4} - \alpha + \left(1 + \frac{9}{2k^2} \right) \alpha^2$. One can check that p is an upward opening parabola which is minimized at the point

$$\alpha^* = \frac{k^2}{2k^2 + 9}.$$

Also, one can check that

$$p(\alpha^*) = \frac{9}{4(2k^2 + 9)} > 0.$$

This seems good, but there is a problem. The problem is that $p(\alpha^*) \rightarrow 0$ as $k \rightarrow \infty$. The reason that this is (potentially) a problem is because we estimated C_2 and C_3 in equations (10.6) and (10.7) for k large, and then used these estimates for φ and φ' . It is possible that we could have $p(\alpha^*) < 0$ when we plug in the actual values for C_2 and C_3 .

We fix this as follows. We may consider the components of the curvature tensor R_{ijkl} as functions of the independent variables e_ε , α , and k , where $e_\varepsilon, k > 0$ and $0 \leq \alpha \leq k$. The first five inequalities are all satisfied for all values of α if ε is sufficiently small and k is sufficiently large. So there exists $\varepsilon_1, k_1 > 0$ such that these inequalities hold for all $\varepsilon \leq \varepsilon_1$ and $k \geq k_1$.

Notice that $\alpha^* \rightarrow \frac{1}{2}$ as $k \rightarrow \infty$. Set $\alpha = \frac{1}{2}$ in the first five inequalities (inequalities (1b) through (3b)) and fix $\varepsilon \leq \varepsilon_1$. For each $k \geq k_1$ let $\delta_k > 0$ be the small positive constant by which inequality (1a) is permitted to fail (whose existence is guaranteed by Corollary 9.5). Since inequalities (1b) through (3b) all hold as $k \rightarrow \infty$, there exists some corresponding $\delta_\infty > 0$ also guaranteed by Corollary 9.5. Clearly, $\delta_k \rightarrow \delta_\infty$ as $k \rightarrow \infty$.

Since $p(\alpha^*) \rightarrow 0$ as $k \rightarrow \infty$, there exists $k_2 > 0$ such that inequality (1a) fails by at most $\delta_k/2$ for all $k \geq k_2$ in a sufficiently small neighborhood of $\alpha = \frac{1}{2}$. So we choose $k = k_2$, and then choose $\varepsilon > 0$ small enough so that $(k_2 + 1)e_\varepsilon$ is less than $\frac{1}{4}$ of the normal injectivity radius of S . This completes the proof of Theorem 1.1. \square

12. CONSTRUCTING AN A -REGULAR METRIC WITH NEGATIVE SECTIONAL CURVATURE

Following [Bel11], let us first recall the definition of an A -regular metric. A Riemannian metric is called A -regular if there exists a sequence of positive numbers $\{A_k\}_{k=0}^\infty$ such that, for each k , the k^{th} covariant derivative of the curvature tensor satisfies the relation $\|\nabla^k R\|_{C^0} < A_k$. Note that, as a consequence when $k = 0$, we have that the sectional curvature is bounded from both above and below. Two facts that are very relevant to our situation are that any metric on a compact manifold is A -regular, and that a locally symmetric metric is A -regular. So in order to modify the metric from Section 10 to make it A -regular, we only need to worry about a small neighborhood of the cusp(s) that were created when we drilled out S . In particular, we modify the metric of Section 10 over a region which is to the left of a_ε , so that $h_\theta = h_r$ and we can use the formulas from Section 8.

For completeness, we provide the details below for how we alter the metric from Section 10. For this we are basically copying the beginning of Section 11 of [Bel11]. But in order to prove that this metric is A -regular with negative sectional curvature, we simply explain why Belegarde's argument in [Bel11] goes through in our setting virtually verbatim.

Let a_ε be as in Section 10, and let $\tau_\varepsilon := \varepsilon e^{a_\varepsilon}$. Notice that $0 < \tau_\varepsilon < 2\varepsilon$ since e^{a_ε} is clearly less than 2. Then let $o_\varepsilon := \ln(\tau_\varepsilon)$ and $p_\varepsilon := 2\ln(\tau_\varepsilon)$. Both o_ε and p_ε are negative and approach $-\infty$ as $\varepsilon \rightarrow 0$. Also, $p_\varepsilon < o_\varepsilon = a_\varepsilon + \ln(\varepsilon) < a_\varepsilon$. Define

$$F(r) := \frac{1}{2} \cdot \frac{e^{\frac{r}{2}}}{\tau_\varepsilon + e^{\frac{r}{2}}} = [\ln(\tau_\varepsilon + e^{\frac{r}{2}})]'.$$

Note that $F' > 0$, $F \in (0, \frac{1}{2})$, and $F(p_\varepsilon) = \frac{1}{4}$.

The following is just a restatement of Proposition 11.1 of [Bel11].

Proposition 12.1. *For all $\varepsilon > 0$ there exists a C^1 function g such that*

- *the function g is positive and increasing.*
- *if $r \geq o_\varepsilon$ then g coincides with the function h from Proposition 10.3. In particular, $g(r) = e^{\frac{r}{2}}$ for $r \in [o_\varepsilon, o_\varepsilon + 1]$.*
- *we have that $g(r) = \tau_\varepsilon + e^{\frac{r}{2}}$ for $r \in (-\infty, p_\varepsilon]$.*
- *if $r \in [p_\varepsilon, o_\varepsilon]$ then g is smooth, $\frac{g'}{g}$ is increasing, $\frac{g'}{g} \in [\frac{1}{4}, \frac{1}{2}]$, and $\frac{g''}{g} > \left(\frac{g'}{g}\right)^2 \geq \frac{1}{16}$.*

As in Section 10 we can use Lemma 10.2 to smooth out g in arbitrarily small regions near p_ε and o_ε while controlling g , g' , and g'' . Now, let γ denote the Riemannian metric constructed in Section 10, but with the warping function h replaced by g . We then have the following Theorem (which is Theorem 11.3 in [Bel11]):

Theorem 12.2. *For $\varepsilon > 0$ sufficiently small the metric γ is A -regular, and can be smoothed near p_ε and o_ε via Lemma 10.2 so that its sectional curvatures are strictly less than 0.*

The proof in [Bel11] works in the present setting as well. All we do now is explain why this is so in the few spots where our formulas differ.

First, Belegradek uses an induction argument to show that γ is A-regular. His argument for the base case goes through verbatim in our setting. In the induction step, the covariant derivatives that arise in [Bel11] are slightly different than what come up here (Theorem 6.1). But we still have that $\nabla_{Y_4} Y_i = 0$ for $i = 1, 2, 3$ (and recall that for us $Y_4 = \frac{\partial}{\partial r}$, but in [Bel11] $\frac{\partial}{\partial r} = Y_0$), and so his arguments as to why $\nabla^{k+1} R$ is bounded still apply.

Belegradek then shows that the sectional curvature of γ is strictly less than zero. He shows this by breaking the region $(-\infty, o_\varepsilon + \sigma)$, where σ is a small constant from Lemma 10.2, up into three different parts. But by applying Lemma 9.4 we only need to consider two regions: $(-\infty, p_\varepsilon)$ and $(p_\varepsilon, o_\varepsilon)$. These regions correspond to Steps 2 and 3 in Section 11 of [Bel11].

The only relevant formulas for our sectional curvature tensor that differ from those in [Bel11] are for R_{2323} and R_{1423} . But the bounds used in [Bel11] still work in our setting. In particular, we also have that

$$R_{2323} < -\left(\frac{g'}{g}\right)^2 \quad \text{and} \quad |R_{1423}| < \frac{v}{2h^2} \left(\ln \frac{v}{h}\right)'.$$

Thus, the arguments in [Bel11] show that γ has negative curvature, proving Theorem 12.2. □

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